Spaces of Constant Curvature

Anthony Salib
Supervisor: Julie Clutterbuck

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1 Introduction

In Geometry of Surfaces [2], Stillwell begins in chapter 1 by introducing the Euclidean Plane and the isometries of the plane, then in chapter 2 defines Euclidean surfaces as specific quotients of the Euclidean plane. In Chapter 3 and 4, he introduces the sphere and the hyperbolic plane, what lines look like in these spaces and then what the isometries are of these spaces. Then in a similar manner to Chapter 2, defines spherical and hyperbolic surfaces as specific quotients of these spaces.

Following this format, the project goals was to formalise the Euclidean, Spherical and Hyperbolic Spaces in n-dimensions, the notion of geodesics (lines) in these spaces and discuss the curvature of these spaces using the analytic techniques of Riemannian geometry. Sections 2-4 of this report will detail this procedure and the results achieved.

After we have introduced the spaces that were dealt with in Stillwell, we will then study their curvature in section 5 and indeed show that these are spaces of constant curvature. Finally in section 6, we will extend the classification results that Stillwell presents for Euclidean, Spherical and Hyperbolic surfaces in 2 and 3 dimensions to n-dimensions.

2 Euclidean

Euclidean space, $\mathbb{R}^n$ can be covered with a single chart, $(\mathbb{R}^n, Id)$. Evidently, $\mathbb{R}^n$ is a smooth manifold, as there are no chart transition functions. The metric is the Euclidean metric,

$$g = \delta_{ij} dx^i \otimes dx^j \tag{1}$$

and so the Christoffel Symbols for the Riemannian Connection are all

$$\Gamma^i_{jk} = 0. \tag{2}$$

To determine what the geodesics look like, take some smooth curve $\gamma : I \to \mathbb{R}^n$ and require that the covariant derivative of it's velocity along the curve vanishes; the requirement that it is a geodesic. We first expand the velocity out into a co-ordinate basis

$$\dot{\gamma}(t) = \dot{\gamma}^j(t)\partial_j,$$

and now we take the covariant derivative,

$$\begin{aligned}
D_t\dot{\gamma} &= \ddot{\gamma}^j(t)\partial_j + \dot{\gamma}^j(t)\nabla_{\dot{\gamma(t)}^i\partial_i}\partial_j \\
&= \ddot{\gamma}^j(t)\partial_j + \dot{\gamma}^j(t)\dot{\gamma}^i(t)\nabla_{\partial_i}\partial_j \\
&= \ddot{\gamma}^j(t)\partial_j + \dot{\gamma}^j(t)\dot{\gamma}^i(t)\Gamma^k_{ij}\partial_k \\
&= \ddot{\gamma}^j(t)\partial_j.
\end{aligned}$$

Now the requirement that $D_t\dot{\gamma} = 0$ gives the second order ODE

$$\ddot{\gamma}^j(t) = 0, \quad j = 1, \dots, n. \tag{3}$$

Hence, the geodesics in $\mathbb{R}^n$ are the constant speed parametrisations of straight lines where

$$\gamma(t) = \alpha t + \beta t \quad \alpha, \beta \in \mathbb{R}^n.$$

3 Spherical

Turning our attention to the n-dimensional spheres of radius 1, S^n, defined by

$$S^n = \{x \in \mathbb{R}^{n+1} : \|x\| \leq 1\}. \tag{4}$$

The first task is prove that it is a smooth manifold.

First, denote the north pole as $N = (0, ..., 1)$ then we choose open sets

$$U_N = S^n \backslash \{N\}$$

and

$$U_S = S^n \backslash \{-N\}.$$

Take a point $x = (x^1, ..., x^n, x^{n+1}) \in \mathbb{S}^n$, and let $u = (x^1, ..., x^n)$ and $\tau = x^{n+1}$, and so we can write a point $x \in \mathbb{S}^n$ as $x = (u, \tau)$. We define the stereographic projection from the north pole to be

$$\phi_N(x) = \phi_N(u, \tau) = \frac{1}{1-\tau}u, \tag{5}$$

and it's inverse will be

$$\phi_N^{-1}(x) = \frac{1}{1+|x|^2}(2x, |x|^2 - 1). \tag{6}$$

We denote the stereographic projection through the south pole to be

$$\phi_S(x) = \phi_N(-x).$$

It follows from 5 and 6 that $\phi_N : U_N \to \phi(U_N)$ is a homeomorphism, and hence the same is true for $\phi_S = \phi_N \circ (-Id)$ from U_S onto $\phi(U_S)$.

The atlas

$$A = \{(U_N, \phi_N), (U_S, \phi_S)\},$$

covers S^n and since on $U_N \cap U_S$

$$\phi_S^{-1} \circ \phi_N(x)$$

and

$$\phi_N^{-1} \circ \phi_S(x)$$

are smooth, S^n is a smooth manifold.

We now wish to define a metric, g_{round}, for the n-sphere. We will do this by pulling back the Euclidean metric on $\mathbb{R}^{n+1}$ to S^n using the map ϕ_N as follows.

$$\phi_N^* g = \delta_{ij} d\big(\frac{x^i}{1-x^{n+1}}\big) \otimes d\big(\frac{x^j}{1-x^{n+1}}\big) \tag{7}$$

$$= \sum_{i=1}^{n} \big(\frac{-x^i dx^{n+1}}{(1-x^{n+1})^2} + \frac{dx^i}{1-x^{n+1}}\big)^2 \tag{8}$$

This will give a metric for all points excluding the north pole, however, doing the same calculation using the map ϕ_S will yield a metric for the north pole which is the same 8. We now turn to the question of what are the geodesics of S^n given g_round.

Now, in the case of the n-spheres and the hyperbolic spaces that will follow, calculating what the geodesics are straight from the covariant derivative is a tedious task. Instead, here we take the claim of Stillwell that the geodesics, (or lines as he calls them), will be given by intersections of the 2-sphere with planes running through the origin, known as great circles, and generalise it to n-dimensions. Hence, the geodesics on S^n will be the curves given by the intersection of 2-planes that run through the origin (planes defined in two dimensions) with S^n, also known as great circles. A symmetry argument for this in n-dimensions is presented in [1].

4 Hyperbolic Space

There are many different models of hyperbolic space, the most natural image that comes to mind is the upper sheet of the hyperboloid. However, Stillwell does not mention this at all, but rather introduces the Poincare upper half space model and then shows that it is isometric to the Poincare disk model in sections 4.1 and 4.2. In this section, we will first introduce the hyperboloid model and show that it is a smooth manifold. We then outline the two models of hyperbolic space in n-dimensions presented by Stillwell, then we will show that the hyperboloid model is isometric to these and so is consistent with the results presented in Stillwell. We will then look at the geodesics in each of the three models, extending the work of Stillwell in Chapter 4.

To simplify the equations that follow, we represent a point $x \in \mathbb{R}^{n+1}$ as $x = (u, \tau)$ where $u = (x^1, \ldots, x^n)$ and $\tau = x^{n+1}$. Now, the hyperboloid model of hyperbolic space is

$$\mathbb{H}^n = \{(u, \tau) \in \mathbb{R}^{n+1} : \tau^2 - |u|^2 = 1, \tau > 0.\}$$

To define a metric on this space, we pull back the minkowski metric,

$$m = (du^1)^2 + \cdots + (du^n)^2 - (d\tau)^2 \tag{9}$$

using the inclusion

$$i : \mathbb{H}^n \to \mathbb{R}^{n+1}.$$

We now prove that it is a smooth manifold. We first note that it can be covered with a single chart $(\mathbb{H}^n, \pi)$ where π is the stereographic projection through $-N$ defined as

$$\pi(u, \tau) = \frac{1}{1+\tau} u \tag{10}$$

and it's inverse is

$$\pi^{-1}(x) = \frac{1}{1 - |x|^2}(2x, |x|^2 + 1). \tag{11}$$

Since there is only one chart and hence no chart transition functions to compute, $\mathbb{H}^n$ is a smooth manifold.

Here, we introduce the other models for the hyperbolic space that Stillwell presented.

The Poincare half space model takes

$$\mathbb{H}^n = \{(x^1, \dots, x^{n-1}, y) \in \mathbb{R}^n : y > 0\}$$

which is the upper half plane, with the metric

$$h_u = \frac{(dx^1)^2 + \cdots + (dx^{n-1})^2 + dy^2}{y^2}.$$

The Poincare ball model is the unit ball B^n with the metric given by

$$h_d = 4\frac{(dx^1)^2 + \cdots + (dx^n)^2}{(1 - |x|^2)^2}.$$

Now, it just so happens that the stereographic projection map π is mapped onto the unit ball B^n. We can see this from the fact that

$$|u|^2 = (\tau^2 - 1) = (\tau - 1)(\tau + 1)$$

and since $\tau > 1$,

$$|\pi(u, \tau)|^2 = \frac{\tau - 1}{\tau + 1} \leq 1.$$

Since 10 and 11 are smooth away from $|x| = 1$, we can conclude that the hyperboloid model is diffeomorphic to the ball model. To show that they are equivalent, we need to show that π is an isometry. s

Definition 4.1 *Let M and M' be two riemannian manifolds and $\phi : M \to M'$ a diffeomorphism. If*

$$\phi^* g' = g,$$

then ϕ is an isometry.

We will consider the map

$$\pi^{-1} : \mathbb{B}^n \to \mathbb{H}^n$$

and we will pull back the Minkowski metric to the unit ball. Our approach will be different to the one we used when calculating the round metric, we will make use of the push forward, π^{-1}_*. Now the push forward will push vectors $V \in T_p\mathbb{B}^n$ to vectors in $\mathbb{H}^n$. Noting that in some co-ordinate chart $\pi^{-1}(x) = (u(x), \tau(x))$ and expanding $V = V^i \partial_i$ we can write the push-forward to be

$$\pi_*^{-1}V = V^i \frac{\partial u^j}{\partial x^i}\frac{\partial}{\partial u^j} + V^i \frac{\partial \tau}{\partial x^i}\frac{\partial}{\partial \tau} = Vu^j \frac{\partial}{\partial u^j} + V\tau \frac{\partial}{\partial \tau},$$

and so we can define the pull back of the minkowski metric on $\mathbb{H}^n$ to $\mathbb{B}^n$ to be

$$(\pi^{-1})^* m(V,V) = m(\pi_*^{-1}V, \pi_*^{-1}V),$$

which is just the minkowski metric evaluated on vectors in the ball model.

Now,

$$Vu^j = \frac{2V^j}{1-|x|^2} + \frac{4x^j < V, x >}{(1-|x|^2)^2}, \tag{12}$$

and

$$V\tau = \frac{4 < V, x >}{(1-|x|^2)^2}. \tag{13}$$

Hence,

$$\begin{aligned} m(\pi_*^{-1}V, \pi_*^{-1}V) &= \sum_{j=1}^{n} (Vu^i)^2 - (V\tau)^2 \\ &= \sum_{j=1}^{n} \left(\frac{2V^j}{1-|x|^2} + \frac{4x^j < V, x >}{(1-|x|^2)^2}\right)^2 - \left(\frac{4 < V, x >}{(1-|x|^2)^2}\right)^2 \\ &= 4\frac{(dx^1)^2 + \cdots + (dx^n)^2}{(1-|x|^2)^2} \\ &= h_d. \end{aligned}$$

Therefore, the hyperboloid model and the half space model are isometric.

In section 4.2, Stillwell shows that the Poincare ball model is isometric to the upper half space model in 2 dimensions. We will not prove it here in n-dimensions, but we note that the same inversion used by Stillwell,

$$J(z) = -i\frac{z+i}{z-i}, \tag{14}$$

can be used in any dimension and so the proof for n dimensions is the same as Stillwell presents, with some minor modifications. So, we conclude that the hyperbolic model presented here is the same as the two models that Stillwell presented in sections 4.1 and 4.2.

Let's now look at what geodesics look like in each of these models. Beginning with the hyperboloid model, we note in a similar fashion to geodesics of the spheres, the geodesics to be intersection of 2-planes that run through the origin with the upper sheet, known as great hyperbolas.

For the ball model, the geodesics are the lines and arcs that are orthogonal to the boundary of the ball. Finally, for the upper half space model, we the geodesics are vertical lines and semi-circles centred on the $y = 0$ axis. These results are proved in n-dimensions in [1]. For $n = 2$, these geodesics can be drawn on a diagram as shown in figure 1.

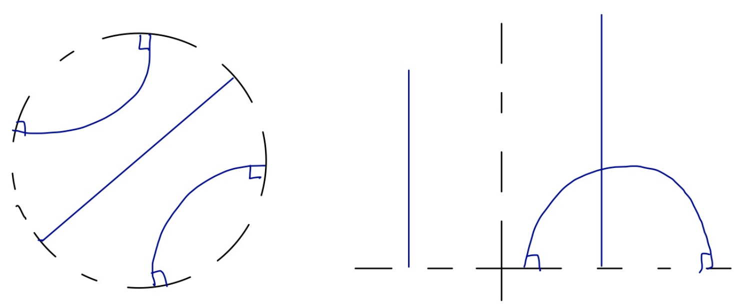

Figure 1: The geodesics (in blue) of the disk model shown on the left and the geodesics of the upper half plane model on the right

5 Curvature

In chapter 4 of Stillwell, he uses the radii of curvature of two normal sections to the pseudo sphere in order to compute the Gaussian curvature. This was quite simple using a diagram for surfaces, but, we would like to extend notions of curvature to higher dimensional manifolds. In this section, we briefly outline the definitions of certain curvature tensors and what they mean geometrically. Then we will use these to show that indeed the spaces introduced above are constant curvature.

Given a connection ∇, the flatness criteria, which is the criteria for covariant differentiation to commute for vector fields X,Y and Z, is given by

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z = \nabla_{[X,Y]} Z, \tag{15}$$

and if satisfied means that manifold with the connection ∇ on it is flat. Knowing this, we would like to define the Riemann Curvature endomorphism to be

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{16}$$

which we can think of as how much our connection deviates from the flatness criterion, i.e. how much our space curves. We then define the Riemann Curvature tensor for vector fields X,Y,Z and W as

$$Rm(X,Y,Z,W) =< R(X,Y)Z, W >, \tag{17}$$

and is invariant under local isometries. This can be represented with lowered indices as

$$R_{ijkl},$$

which is useful notation to define the Ricci Curvature tensor as

$$R_{ij} = g^{kl} R_{kijm} \tag{18}$$

and the Ricci scalar curvature

$$S = g^{ij} R_{ij}. \tag{19}$$

Now that we have this machinery lined up for us, we can now calculate the same curvature as Stillwell did in section 4.1, which he did by

- Picking a plane that is normal to the surface (that contains the normal vector to the surface), call this a normal plane
- Taking the curve that is the intersection of this plane with the surface
- Finding the curvature of this curve, call it κ
- Repeating the above steps for every normal plane

- Then taking the product of the maximum curvature κ_1 and minimum curvature κ_2, these are called the principal curvatures
- $K = \kappa_1\kappa_2$ is the Gaussian curvature.

Now, Gaussian Curvature is the curvature of surfaces in $\mathbb{R}^3$, we need a way to relate what Stillwell did with Gaussian curvature to our generalisation of these spaces to $\mathbb{R}^n$. The way we do this for a riemannian manifold M at some point p is

- Take a 2 dimensional subspace of T_pM, Π
- Take the geodesics which lie in Π and let S be the surface swept out by them
- This is called the plane section
- We can use Gauss' Theorema Egregium to compute the Gaussian curvature of S,

$$K(X, Y) = \frac{Rm(X, Y, Y, X)}{|X|^2|Y|^2 - < X, Y >^2} \tag{20}$$

  where (X, Y) is a basis for Π.
- This is called the sectional curvature of M associated with Π at p

If the sectional curvature of a manifold M is the same for all $p \in M$, then we say that M is of constant curvature.

Now, that we have this, and the Riemann curvature tensor defined, we will compute the sectional curvatures of our spaces. We begin with $\mathbb{R}^n$, which is the easiest case, since it is flat, and so the curvature tensor is zero for all points. Hence the sectional curvature is zero.

Now we look at S^n which is homogeneous and isotropic, and so we only need to look at the curvature at the north pole using a plane Π spanned by any two basis vectors, so let's choose (∂_1, ∂_2). The geodesics that have initial velocity in Π are great circles which will always remain in the subspace (x^1, x^2, x^{n+1}). So we effectively have circles of radius 1 in $\mathbb{R}^3$. The span of these circles will be S^2. We know from early differential geometry that the the unit sphere has Gaussian curvature 1 and so S^n has constant sectional curvature 1. If we allow S^n to be of any radius R, then it will have constant sectional curvature $\frac{1}{R^2}$, denote this space as S^n_R.

A similar argument yields the same result as Stillwell achieved for $\mathbb{H}^n$, which has constant sectional curvature -1. If we allow the Ball Model of the hyperbolic space to have any radius, we get that $\mathbb{H}^n_R$ will have constant negative sectional curvature $-\frac{1}{R^2}$.

6 The Killing-Hopf Theorem

Stillwell dedicates all of chapter 2, 3.6 and chapter 5 to determine what spaces look like the Euclidean plane, the 2-sphere and the hyperbolic plane. In chapter 2 he showed that each complete, connected euclidean surface is of the form $\mathbb{R}^2/\Gamma$ where Γ is a discontinuous fixed point free group of isometries. He then mirrors this argument for spherical surfaces in section 3.6 and then again for hyperbolic surfaces in chapter 5. Armed with the generalisation of these spaces to n dimensions and radii R, we now introduce the necessary concepts to establish the Killing-Hopf theorem that Stillwell proves for the n=2 case. We begin with the following theorem from [3].

Theorem 6.1 *Let N be a simply connected complete pseudo-riemannian manifold of constant sectional curvature K. Let M be a pseudo-riemannian manifold which has a tangent space isometric to a tangent space of M. Then, M is complete and of constant curvature K iff M is isometric to the quotient N/Γ of N by a properly discontinuous group of isometries.*

With this theorem, we would be able to classify every space of constant curvature as either a quotient of $\mathbb{R}^n$, $\mathbb{S}^n_R$ or $\mathbb{H}^n_R$ if we knew what the groups Γ are. This is precisely the substance of the killing-hopf theorem, but before we can state it we will need to identify the isometries in each of these spaces that act freely and discontinuously. For a group to act freely there must be no fixed points and for it to act discontinuously requires that for any $P \in M$, there must not be a Γ-orbit with a limit point.

Now, what are the possible groups of isometries? We introduce the group of isometries of Euclidean space as $E(n)$ which contains both translations and rotations, and the orthogonal group $O(n) \subset E(n)$ are the isometries of S^{n-1}. The isometries of hyperbolic space $\mathbb{H}^{n-1}_R$ we will denote $O^1(n)$

We now state the killing-hopf theorem.

Theorem 6.2 *Let M be a riemannian manifold of dimension $n \geq 2$ and K a real number. Then M is complete, connected and of constant curvature K iff it is isometric to*

- S^n_R/Γ *where* $\Gamma \subset O(n+1)$, *if* $K > 0$
- $\mathbb{R}^n/\Gamma$ *where* $\Gamma \subset E(n)$, *if* $K = 0$
- $\mathbb{H}^n_R/\Gamma$ *where* $\Gamma \subset O^1(n+1)$, *if* $K < 0$

where Γ acts freely and discontinuously.

7 Conclusion

The results that were presented and developed in John Stillwell's Geometry of Surfaces, for 2 dimensional manifolds, were extended into n dimensions. Specifically, we have formalised the $\mathbb{R}^n$, S^n and $\mathbb{H}^n$ spaced and have looked at their geodesics as well as studied their curvature and shown that they are indeed constant curvature manifolds. We have also shown much of the analytic tools of Riemannian geometry and applied them to these spaces of constant curvature. We have then extended very briefly Stillwell's search of complete, connected and constant curvature manifolds by giving the Killing-Hopf theorem in n-dimensions.

References

[1] J. Lee. *Riemannian Manifolds.* Springer, 1997.

[2] J. Stillwell. *Geometry of Surfaces.* Springer, 1992.

[3] J. Wolf. *Spaces of Constant Curvature.* AMS, 2011.