The Cauchy Problem in General Relativity

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General Relativity

General Relativity studies 4 dimensional Lorentzian manifolds (M, g) that satisfy the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} \tag{1}$$

where
$R_{\mu\nu}$ is the Ricci curvature tensor of the metric g
R is the Ricci Scalar defined as $R := R_{\mu\nu} g^{\mu\nu}$
$T_{\mu\nu}$ is the energy momentum tensor of matter

The analogue to this in Newtonian gravity is the Poisson equation

$$\nabla^2 \psi = 4\pi G \rho \tag{2}$$

Models of Matter

- The vacuum equations, $T_{\mu\nu} = 0$
- Perfect Fluid, $T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu}$
- Einstein-Maxwell Equations,
  $T_{\mu\nu} = \frac{1}{4\pi}(F^{\alpha}{}_{\mu} F_{\nu\alpha} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta})$

The Cauchy Problem

The Einstein Vacuum equations, where we take $T = 0$, reduce to

$$R_{\mu\nu} = 0. \tag{3}$$

The Cauchy problem is concerned with given initial data, can we find a stable solution to 3, "maximal Cauchy devlopment."

The rest of this talk

- Harmonic Gauge
- Well-posedness of the problem

The Harmonic Gauge

De-Donder (1920) found the following Harmonic Gauge

$$\Box x^{\alpha} = 0, \tag{4}$$

in which the Einstein Vacuum equations are of the form

$$\Box_g g_{\mu\nu} = N_{\mu\nu}(g, \partial g), \tag{5}$$

where

$$\Box_g = g^{\mu\nu} \partial^2_{\mu\nu}$$

and are known as the reduced Einstein equation.

Initial Data

- A metric, $\bar{g}$
- The induced second fundamental form of g, K
- Both defined on a 3-manifold (hypersurface), Σ

Moreover, these initial conditions, $(\Sigma, \bar{g}, K)$, must satisfy the vacuum constraint equations

$$\bar{R} + |K^2| + (tr(K))^2 = 0 \tag{6}$$

and

$$\nabla^j K_{ij} - \nabla_i tr(K) = 0. \tag{7}$$

Choquet-Bruhat (1952) - Local Existence

Now we ask, when do solutions of the reduced Einstein equations 5 give solutions of the Einstein equations 3?
The answer is when the solution satisfies the harmonic gauge condition.
However, Choquet-Bruhat found that the harmonic gauge is automatically satisfied given that it is satisfied on Σ and that the initial conditions satisfy the constraint equations. This means choosing a local co-ordinate system that satisfies the harmonic gauge will also give us solutions to the Einstein Equations.

Choquet-Bruhat and Geroch (1969) - Global Existence

For some given initial data, there exists a unique, maximal, globally hyperbolic, development, (M, g). We call this the maximal Cauchy development.

(Siberski, 2013) Proved the same theorem without appealing to Zorn's lemma.

The maximal Cauchy development of the data set $(\Sigma, \bar{g}, K)$ is a central object in the study of general relativity, as it allows us to formulate questions of dynamics as in the Newtonian case.

Questions