General Relativity

Anthony Salib

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Contents

1 Mathematics

This section is a very brief outline of the mathematics required to study general relativity. Schutz [6] and D'Inverno [3] were the primary sources used for the following mathematical exposition.

1.1 Manifolds

Before we give the definition of a manifold, let us motivate it with an example. Begin with the sphere in $\mathbb{R}^3$ and say we want to analyse a temperature function on it, i.e. a function $f : \mathcal{M} \to \mathbb{R}$, how would we be able to define differentiability of this function on the sphere? You may have an intuitive idea of how to do this, but in order to be precise, we must introduce the machinery of topological manifolds and differentiable manifolds. The idea however, is that we want recast everything on the manifold to euclidean space in which we already have the notion of differentiability.

Definition 1.1 *A topological space* $(\mathcal{M}, \mathcal{O})$ *is called an n-manifold if for every point* $p \in M$*, there exists an open neighbourhood* U *and a homeomorphism* $x : U \to x(U) \subset \mathbb{R}^n$*.*

The homeomorphism x is known as a co-ordinate chart and the set of co-ordinate charts that cover the manifold is known as the atlas $\mathcal{A}$. Let's consider a manifold M that is covered with more than 1 chart, so A has at least two charts, $x : U \to x(U)$ and $y : V \to y(V)$. Now, if $U \cap V \neq \emptyset$, then we can represent a point $p \in \mathbb{R}^m$ in two different chart representations. We can transition between them according figure 1, where

$$y \circ x^{-1} \tag{1}$$

is the chart transition map.

Two overlapping charts, x, y are called smoothly compatible if $x \circ y^{-1}$ and it's inverse are smooth, this is known as a diffeomorphism. If in an atlas $\mathcal{A}$ every chart chart transition map is smoothly compatible then $\mathcal{A}$ is called a smooth atlas and $\mathcal{M}$ is called a smooth manifold.

To any smooth manifold M, we can define the tangent space at a point p, T_pM to be the set of all tangent vectors that run through p. Taking any smooth curve γ through the point p and some smooth function f, the tangent vector to γ at p is the map $X_p : C^\infty(M) \to \mathbb{R}$, where

$$X_p(f) = \frac{d}{dt}(f \circ \gamma)(t). \tag{2}$$

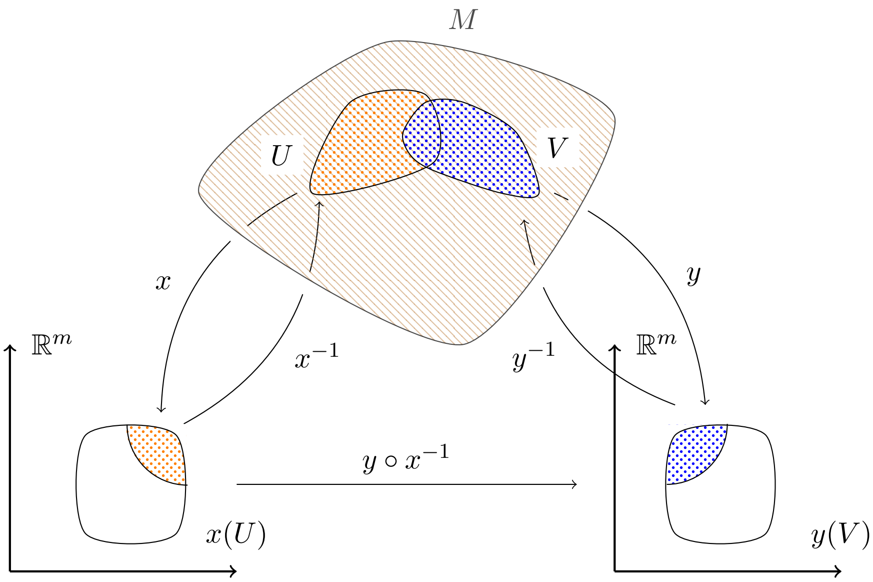


Figure 1: The transition map between two different charts

The set of all these tangent vectors at p is T_pM. We stress that these tangent vectors are maps that are set by some smooth curve γ and take as an argument a smooth function f. This is similar to how a vector in $\mathbb{R}^3$ defines a directional derivative. A basis for T_pM is found by selecting a chart, $x = (x^1, ..., x^n)$, and then the basis is the derivatives

$$\{\frac{\partial}{\partial x^\alpha}\}_p \tag{3}$$

and then an arbitrary tangent vector can be written in the form

$$X^\alpha \frac{\partial}{\partial x^\alpha} \tag{4}$$

and then acts on a function f as

$$X(f) = X^\alpha \frac{\partial f}{\partial x^\alpha}. \tag{5}$$

We note that we have used the Einstein summation convention where a raised and lowered index is summed over. Now again, we can have two different overlapping charts x and y, and so the transformation of a tangent vector comes down to transforming between the chart dependent bases. The basis vectors are transformed according to

$$\Big(\frac{\partial}{\partial x^{\alpha}}\Big)_p = \Big(\frac{\partial}{\partial x^{\beta}}\Big)_p \Big(\frac{\partial y^{\beta}}{\partial x^{\alpha}}\Big)_{x(p)}. \tag{6}$$

Given some vector $X_{(x)} = X^{\alpha}_{(x)} \frac{\partial}{\partial x^{\alpha}}$ at a point p written in the x basis, we can transform it's components to the y basis, $X_{(y)}$

$$X^{\alpha}_{(y)} = \Big(\frac{\partial y^{\alpha}}{\partial x^{\beta}}\Big)_p X^{\beta}_{(x)}. \tag{7}$$

1.2 Tensors

We now define a contravariant tensor of rank 1 as a set of quantities X^{α} that transform according 7. Note that these are the components of the tangent vectors we introduced, we call these contravariant vectors. A contravariant tensor of rank 2 is a set of quantities $X^{\alpha\beta}$ that transform according

$$X^{\alpha\beta}_{(y)} = \Big(\frac{\partial y^{\alpha}}{\partial x^{\gamma}}\Big)_p \Big(\frac{\partial y^{\beta}}{\partial x^{\sigma}}\Big)_p X^{\gamma\sigma}_{(x)}. \tag{8}$$

We can have higher rank contravariant tensors such as X^{abcd} which is of rank 4, and which transform in a similar manner.

Now we consider quantities that transform according to the inverse of 7, namely a set of quantities X_a that are transformed from the x, basis X, to the y basis X', by

$$X'_a = \Big(\frac{\partial x^{b}}{\partial y^{a}}\Big)_p X_b. \tag{9}$$

These quantities are called covariant tensors of rank 1. Note that contravariant tensors have raised indices and covariant tensors have lowered indices. Introducing tensors in this way by their transformation properties is very useful, and bypasses a lot of mathematics that we will not need for this report.

So to summarise, we have two chart representations of the same point p, x and y. We can compute a transformation matrix

$$[\frac{\partial y^a}{\partial x^b}] = \begin{bmatrix} \frac{\partial y^1}{\partial x^1} & \frac{\partial y^1}{\partial x^2} & \cdots & \frac{\partial y^1}{\partial x^n} \\ \frac{\partial y^2}{\partial x^1} & \frac{\partial y^2}{\partial x^2} & \cdots & \frac{\partial y^2}{\partial x^n} \\ . & & & \\ . & & & \\ . & & & \\ \frac{\partial y^n}{\partial x^1} & \frac{\partial y^n}{\partial x^2} & \cdots & \frac{\partial y^n}{\partial x^n} \end{bmatrix}, \tag{10}$$

and note that contravariant tensors transform according to the matrix and that covariant tensors transform according it's inverse. We note that according to our construction here, a tensor is any quantity that transforms according to these rules.

A contravariant tensor of rank r is an $(r, 0)$ tensor and covariant tensor of rank q is a $(0, q)$ tensor. A mixed (p, q) tensor is p contravariant and q covariant. So, a (1,2) tensor would be X^a_{bc} and would transform according

$$X'^a{}_{bc} = \frac{\partial y^a}{\partial x^d} \frac{\partial x^e}{\partial y^b} \frac{\partial x^f}{\partial y^c} X^d{}_{ef}. \tag{11}$$

A tensor field assigns a tensor to every point in the manifold.

1.3 Metrics

A special type of tensor is the metric, g_{ab}, which is any covariant (0,2) tensor that is symmetric, which means that $g_{ab} = g_{ba}$. The metric defines the line element used on our manifold to measure distances,

$$ds^2 = g_{ab} dx^a dx^b, \tag{12}$$

as well as the length of vectors

$$|X| = \sqrt{g_{ab} X^a X^b}. \tag{13}$$

The determinant of the metric is given the symbol

$$g = det(g) \tag{14}$$

and if $g \neq 0$ then the inverse of the metric, g^{ab}, exists and is defined such that

$$g_{ab}g^{bc} = \delta^c_a. \tag{15}$$

We can use the metric and it's inverse to raise and lower indices such as

$$g_{ij}X^j = X_i, \tag{16}$$

and

$$g^{ij}\omega_i = \omega^j. \tag{17}$$

1.4 Curvature

The metric not only gives information on length between points or the length of vectors, it also tells us how a space is curved. The covariant derivative ∇ which differentiates some vector $Y = Y^j E_j$ in the direction of another vector $X = X^i E_i$ is given by

$$\nabla_X(Y) = (XY^j)E_j + Y^j X^i \Gamma^k{}_{ji} E_k, \tag{18}$$

where $\{E_i\}$ is the basis vectors for the tangent space and the Christoffel Symbol is defined as how the basis vectors differentiate with respect to each other,

$$\Gamma^k{}_{ji} = \nabla_{E_i}(E_j). \tag{19}$$

For example, in Cartesian co-ordinates, moving along one axis produces no change in any of the other axis, and so all Christoffel symbols for this co-ordinate basis is zero. But for polar co-ordinates with basis $\{E_\theta, E_r\}$, this is not the case, and in fact we can show that

$$\frac{\partial E_r}{\partial E_\theta} = \frac{1}{r} E_\theta \tag{20}$$

and so this would mean that

$$\Gamma^\theta{}_{r\theta} = \frac{1}{r}. \tag{21}$$

This is not the only non-zero Christoffel symbol, since the Christoffel symbols are symmetric, we get that

$$\Gamma^{\theta}{}_{\theta r} = \frac{1}{r}, \tag{22}$$

and we can show that

$$\Gamma^{r}{}_{\theta\theta} = -r. \tag{23}$$

We note that the covariant derivative is completely specified by the choice of Christoffel symbols $\Gamma^{k}{}_{ij}$, these can be chosen at will, but in practice, they are determined by the metric

$$\Gamma^{a}{}_{bc} = \frac{1}{2} g^{ad}(\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc}). \tag{24}$$

The Christoffel symbols are dependent on a chart and hence transform according to some transformation law, however, we note that it does not follow the transformation laws for tensors and so the Christoffel symbols are not tensors. However, the difference between two Christoffel symbols does transform according to 10 and so the difference between two Christoffel symbols is a tensor. There exist co-ordinates at each point p in the manifold such that all the Christoffel symbols are zero, these are called normal co-ordinates.

From the Christoffel symbols, we can compute the Riemann curvature tensor

$$R^{i}{}_{jkl} = \partial_k \Gamma^{i}_{lj} - \partial_l \Gamma^{i}_{kj} + \Gamma^{i}_{k\lambda}\Gamma^{\lambda}_{lj} - \Gamma^{i}_{l\lambda}\Gamma^{\lambda}_{kj} \tag{25}$$

which tells us how far from flat our manifold with metric is. From this we can compute the Ricci Curvature tensor as

$$R_{ij} = g^{kl} R_{kijm} \tag{26}$$

and the Ricci scalar to be

$$R = g^{ij} R_{ij}. \tag{27}$$

We can also define geodesics as curves that have minimum curvature which means that its covariant derivative vanishes. This is equivalent to a curve, $x(t)$, satisfying the geodesic equation

$$\ddot{x}^{k}(t) + \dot{x}^{i}(t)\dot{x}^{j}(t)\Gamma^{k}_{ij}(x(t)) = 0. \tag{28}$$

1.5 Integration

We would like to define actions on our spacetime manifold, and to do that we need to be able to integrate over the manifold. To this end, we need to add into our integral a volume form, and again, this is given by the metric, specifically the determinant of it g, such that integrals are

$$\int_M f d^4x = \int_M d^4x\sqrt{-g}f. \tag{29}$$

Now that all the mathematical pre-requisites have been covered, we begin delving into gravity. First we will look at Newtonian spacetime and show that the idea of curved spacetime is not only realised in general relativity but is indeed a consequence of Newtons laws.

2 Newtonian Spacetime

We begin by stating Newton's laws and making two key observations.

Theorem 2.1 ***Newton's Laws***
Newton I: A body on which no force acts moves uniformly along a straight line.
Newton II: Any deviation from straight uniform motion is effected by a force.

Now the first law is just a special case of the second law, but, we must argue that Newton wasn't naive and that the first law is there to serve a deeper purpose than only to be a "special case." Indeed, in order for it to be relevant, the first law speaks of the geometry of space, in particular, how can we measure physical geodesics.

The second observation we make is that if we consider gravity a force, in our universe filled with matter, Newtons first law does not apply, as there will be gravitational forces acting on objects at all times. Hence, in order for Newtons first law to remain applicable, we must not consider gravity a force.

So, what is gravity? Equipped with our knowledge of differential geometry and geodesics, we first ask, can gravity be encoded in the curvature of space? A particle of mass m will experience an acceleration due to the gravitational field due to some mass density ρ given by

$$m\ddot{x}^\alpha(t) = mf^\alpha(x(t)) \tag{30}$$

where f is from Poisons equations

$$-\partial_\alpha f^\alpha = 4\pi G\rho. \tag{31}$$

By the weak equivalence principle which states that inertial mass is equivalent to gravitational mass, we can deduce that

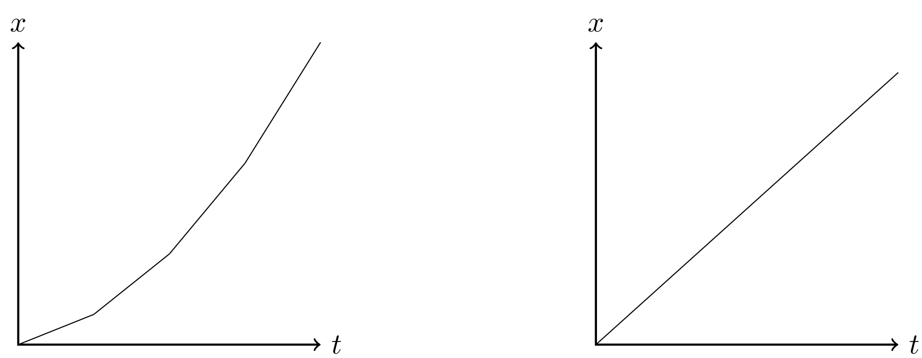
$$\ddot{x}^\alpha(t) = f^\alpha(x(t)) \tag{32}$$

and so

$$\ddot{x}^\alpha(t) - f^\alpha(x(t)) = 0. \tag{33}$$

Now, if gravity were encoded into the geometry of space, then 33 would resemble the geodesic equation 28. However, on close inspection, there is no set of connection coefficients that will allow 33 to take the form of 28.

We now return to the statement of Newtons first law and consider what uniform motion is. Of course, if someone were to ask us what we consider to be uniform motion, we would tell them motion that is at a constant velocity. However, in our attempt above, we have really only tried to enforce the idea of motion in a straight line, there is no way to enforce the notion of constant velocity in our attempt. The key insight that we make here is shown in the following spacetime diagrams.



(a) Non-constant velocity in spacetime

(b) Constant velocity in spacetime

Figure 2

We can see from figure 2b that uniform motion in spacetime is straight line motion, unlike in figure 2a where a variable velocity does not yield motion in a straight line. So, we reattempt what we did previously, but in spacetime, not just space. What do I mean by this?

Well, before we were considering the path of a particle $x : \mathbb{R} \to \mathbb{R}^3$ where $t \mapsto x(t)$, but now we will consider the word line of the particle $X : \mathbb{R} \to \mathbb{R}^4$ where $t \mapsto (t, x^1(t), x^2(t), x^3(t)$. Now, we can easily see that $\dot{X}^0 = 1$, and assuming that $\ddot{x}^\alpha(t) = f^\alpha(x(t))$, we can write the components of $\ddot{X}$ as

$$\ddot{X}^0 = 0 \tag{34}$$
$$\ddot{X}^\alpha - f^\alpha(X(t)) \cdot \dot{X}^0 \cdot \dot{X}^0 = 0 \tag{35}$$

where 35 has the form of the geodesic equation. Hence, we can choose the connection coefficients of Newtonian spacetime to be

$$\Gamma^\alpha_{\ 00} = -f^\alpha \tag{36}$$

and the rest to be zero. So, Newtonian spacetime is indeed curved, and the curvature of spacetime is indeed coded in Newtons laws.

3 General Relativity

In this section, we will derive Einstein's field equations from an action, but before we do that we will introduce actions, and the types of matter that we can represent with actions.

3.1 Matter

Classical field matter is any tensor field on spacetime whose equation of motion derive from an action.

For example, if F is the Faraday tensor defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{37}$$

where A is a (0,1)-tensor known as the 4-potential defined as

$$A = (\phi, \mathbf{A}) \tag{38}$$

where $\mathbf{A}$ is a magnetic potential and ϕ is the electric potential. We can define the action for the electromagnetic field in curved space

$$S_{Maxwell}[A;g] = \frac{1}{4}\int_M d^4x\sqrt{-g}F_{ab}F_{cd}g^{ac}g^{bd} + A(j). \tag{39}$$

With each action we can associate a Lagrangian which is $\mathcal{L}$ such that the action is written as

$$S = \int_M d^4x\mathcal{L}. \tag{40}$$

So, in the case of our example, the Lagrangian for the electromagnetic field is

$$\mathcal{L}_{\mathcal{EM}} = \sqrt{-g}(-\frac{1}{4}F_{ab}F_{cd}g^{ac}g^{bd} + A(j)). \tag{41}$$

Now given a Lagrangian, we define the energy momentum tensor to be

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}}. \tag{42}$$

Hence, the energy-momentum tensor for the electromagnetic field is

$$T_{\mu\nu} = F_{\mu\sigma}F^{\sigma}_{\nu} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}. \tag{43}$$

Another example of a stress energy tensor that can be derived in a similar way is that of a perfect fluid

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + Pg_{\mu\nu}. \tag{44}$$

As we will see in the next section, the Einstein-Hilbert action given by the Lagrangian $\mathcal{L}_{\mathcal{E}-\mathcal{H}}$ will give the Einstein field equations by varying it with respect to metric. So to obtain the Einstein Field equations in any matter distribution, we just need to vary the action given by $\mathcal{L}_{\mathcal{E}-\mathcal{H}}$ coupled with the Lagrangian for the matter distribution. So for example the Einstein-Maxwell equations will be given by varying the action

$$S = \int_M d^4x\mathcal{L}_{\mathcal{E}_{\mathcal{H}}} + \mathcal{L}_{\mathcal{M}}. \tag{45}$$

The addition of the matter field Lagrangian only adds on a stress-energy tensor, so we usually specify the Einstein field equations as

$$R_{ab} - \frac{1}{2}g_{ab}R = 8\pi G T_{ab}. \tag{46}$$

and specify the stress energy tensor T_{ab}.

3.2 The Einstein Field Equations

In this section, we will vary the Einstein-Hilbert action,

$$\mathcal{S}[g] = \int_M \sqrt{-g} R_{ab} g^{ab}, \tag{47}$$

with respect to the metric g_{ab}, and require that this variation is zero as per the principle of least action,

$$\delta \mathcal{S} = 0, \tag{48}$$

which means that

$$\delta \mathcal{S} = \int_M \delta\sqrt{-g} R_{ab} g^{ab} + \sqrt{-g}\delta R_{ab} g^{ab} + \sqrt{-g} R_{ab} \delta g^{ab} = 0. \tag{49}$$

We now compute each term that is present in 49, beginning with

$$\delta\sqrt{-g} = \frac{-det(g) g^{mn} \partial g_{mn}}{2\sqrt{-det(g)}} \tag{50}$$

$$= \frac{1}{2}\sqrt{-det(g)} g^{mn} \partial g_{mn} \tag{51}$$

Omitting the $det(g)$ and just writing g we get that

$$= \frac{1}{2}\sqrt{-g} g^{mn} \partial g_{mn}. \tag{52}$$

We now turn our attention to varying the ricci tensor of the metric. To do this, we re-call the definition of the Ricci Curvature Tensor,

$$R_{ab} = R^{\lambda}_{a\lambda b}, \tag{53}$$

and so we get that we need to vary

$$\partial_m \Gamma^m_{ab} - \partial_a \Gamma^m_{mb} + \Gamma^m_{m\lambda}\Gamma^\lambda_{ab} - \Gamma^m_{a\lambda}\Gamma^\lambda_{mb} \tag{54}$$

We first note that if we choose a normal co-ordinate chart at a point, which is possible for each point, the Christoffel symbols vanish, however, their derivatives remain as it only vanishes at a point. So we must now vary

$$\begin{aligned} \delta R_{ab} &= \delta\partial_b \Gamma^m_{am} - \delta\partial_m \Gamma^m_{ab} && (55) \\ &= \partial_b \delta\Gamma^m_{am} - \partial_m \delta\Gamma^m_{ab}. && (56) \\ & && (57) \end{aligned}$$

We stop here and ponder what does it mean to vary the connection co-efficient Γ. It means that we take two completely different co-coefficients and subtract them. Now, the difference between two different connection co-coefficients is a (1,2)-tensor, which we call $\delta\Gamma$. Now, before we can go back to our calculation, we need to make one more observation, and that is in normal co-ordinates

$$\partial_b \delta\Gamma^m_{am} = \nabla_b (\delta\Gamma)^m_{am}, \tag{58}$$

as the difference between the covariant derivative and the partial derivative are Christoffel symbols, which all vanish in normal co-ordinates. Now we can write that

$$\begin{aligned} \delta R_{ab} &= \nabla_b (\delta\Gamma)^m_{am} - \nabla_m (\delta\Gamma)^m_{ab}, && (59) \\ & && (60) \end{aligned}$$

and so

$$\sqrt{-g} g^{ab} \delta R_{ab} = \sqrt{-g} g^{ab} \nabla_b (\delta\Gamma)^m_{am} - \sqrt{-g} g^{ab} \nabla_m (\delta\Gamma)^m_{ab}, \tag{61}$$

Since $\nabla g = 0$, we can pull the metric inside the covariant derivatives

$$= \sqrt{-g} \nabla_b (g^{ab}\delta\Gamma)^m_{am} - \sqrt{-g} \nabla_m (g^{ab}\delta\Gamma)^m_{ab}, \tag{62}$$

and letting $A^b = g^{ab}\delta\Gamma^m_{am}$ and $B^m = g^{ab}\delta\Gamma^m_{ab}$ we get

$$= \sqrt{-g}\nabla_b A^b - \sqrt{-g}\nabla_m B^m \tag{63}$$

$$= \nabla_b(\sqrt{-g}A^b) - \nabla_m(\sqrt{-g}B^m). \tag{64}$$

where the inequality in the final line is again due to the fact that $\nabla g = 0$. We now compute the final term, δg^{ab}. We first observe that

$$g^{ab}g_{bc} = \delta^a_c, \tag{65}$$

and taking variation of 65 we get that

$$\delta(g^{ab})g_{bc} + g^{ab}\delta(g_{bc}) = 0 \tag{66}$$

which is re-arranged to give

$$\delta(g^{ab}) = -g^{bn}g^{am}\delta g_{mn}. \tag{67}$$

We now put all these terms back into 49 and we get that

$$\int_M \big[\frac{1}{2}\sqrt{-g}g^{mn}\partial g_{mn}g^{ab}R_{ab} - \sqrt{-g}g^{bn}g^{am}\delta g_{mn}R_{ab} + \nabla_b(\sqrt{-g}A^b) - \nabla_m(\sqrt{-g}B^m)\big] \tag{68}$$

By Stokes theorem, the $\nabla_b(\sqrt{-g}A^b) - \nabla_m(\sqrt{-g}B^m)$ terms are integrated to boundary terms. Since the variation of the metric vanishes on the boundary, these boundary terms are zero and so

$$= \int_M \sqrt{-g}\delta g_{mn}\big[\frac{1}{2}g^{mn}R - R^{mn}\big], \tag{69}$$

and since this must be zero, the terms in square brackets must be zero which leads to the Einstein field equations in vacuum

$$R_{ab} - \frac{1}{2}g_{ab}R = 0. \tag{70}$$

If we had chosen a different action, one coupled to some matter field that produces an energy-momentum tensor T_{ab} we would get

$$R_{ab} - \frac{1}{2} g_{ab} R = 8\pi G T_{ab}. \tag{71}$$

These are known as the Einstein Field equations, and represents 10 partial differential equations which can be solved for the metric, which tells us how our spacetime is curved. If instead we choose our action to be

$$\mathcal{S}[g] = \int_M \sqrt{-g} (R_{ab} + 2\Lambda) g^{ab}, \tag{72}$$

where Λ is the cosmological constant, we will get that the Einstein field equations are

$$R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab} = 8\pi G T_{ab}. \tag{73}$$

Here the cosmological term was added by Einstein to force his way to a solution which described a static closed universe. Although we do now know that the universe is expanding, the cosmological term is still kept as an expression for the vacuum energy density of the universe.

Now there are a number of different ways to look at this equation. The first way is to say, we know the matter distribution, T_{ab}, what is the metric? So in this view, we specify the energy momentum tensor and solve for the corresponding geometry of spacetime. The second way to look at the equations are as a set of constraint equations, so that if we can deduce certain parts of the geometry or matter distribution from physical phenomena, then we can use the constraints to determine the rest of the geometry and/or matter distribution.

An important special case of the Einstein equations is when $T_{ab} = 0$, and these are known as the vacuum Einstein equations,

$$R_{ab} - \frac{1}{2} g_{ab} R = 0. \tag{74}$$

This reduces to the condition that the metric is Ricci flat,

$$R_{ab} = 0. \tag{75}$$

3.3 The Cauchy Problem

The Cauchy problem involves given some initial data set, can we find a solution to our set of partial differential equations? In this section we will only consider evolution of initial data under the vacuum field equations 75 and we aim to understand the fundamental theorem of general relativity given by Geroch and Choquet-Bruhat [1] and [2]. The fundamental theorem is as follows.

Theorem 3.1 *Let $(\Sigma, \bar{g}, K)$ be a smooth initial data set. There exists a unique smooth spacetime (M, g) such that*

1. $R_{\mu\nu} = 0$
2. *(M,g) is globally hyperbolic with Cauchy surface Σ, with induced first and second fundamental form $\bar{g}$ and K*
3. *Any other smooth spacetime that has the above properties isometrically embeds into M.*

Now, let's try to understand this. The first thing that we need to understand is what is the initial data, and what does it mean for it to be smooth? The initial data that we need to specify is some 3 dimensional riemannian surface Σ along with the metric on it $\bar{g}$ and the second fundamental form on it K. Now in order for these things to be smooth, they need to satisfy the following constraint equations

$$\bar{R} + |K^2| + (tr(K))^2 = 0 \tag{76}$$

and

$$\nabla^j K_{ij} - \nabla_i tr(K) = 0. \tag{77}$$

Once these are satisfied, the theorem states that this initial data set can be evolved by the vacuum Einstein equations to produce a maximal spacetime (M, g) that is globally hyperbolic. Globally hyperbolic means that the manifold admits a Cauchy surface, which is any subset of spacetime that is intersected once by every inextensible causal curve. The theorem not only says that it is globally hyperbolic, but it also gives the Cauchy surface that (M, g) admits, namely, Σ.

Finally the theorem states that this spacetime is maximal, that if there is some other smooth spacetime out there that admits the first two properties, then it is part of (M, g). Isometrically embedded means that it can be mapped into (M, g) while keeping all information on distance and curvature the same.

Now that we have looked at the Einstein field equations and the various matter distributions that we can couple to them, we will now look at the two simplest special solutions for the vacuum equations, the Minkowski Solution and the Schwarzschild solution.

4 The Minkowski Solution

The simplest solution to the vacuum equations is Minkowski space of special relativity, $(\mathbb{R}^4, g_M)$ with

$$ds_M^2 = -dt^2 + dx^2 + dy^2 + dz^2. \tag{78}$$

Clearly, from 24, all the Christoffel symbols vanish, and so this solves the vacuum Einstein equations. Now, we would like to study this solution, and there is a number of ways to do this. In this report, we are going to look at the causal structure of the spacetime, although there are other ways to study the spacetime such as stability.

We need to first introduce some terminology, and to do that in the following spacetime diagrams, we will work in two dimensional Minkowski space, $(\mathbb{R}^2, g)$ where

$$ds^2 = -dt^2 + dx^2. \tag{79}$$

We first note that light rays are straight lines that have gradient of ± 1, at some point p, this forms a light cone as shown in figure 3.

Now this light cone actually determines what can and cannot send information to p, so an observer at q shown in figure 3 cannot send a signal to p without travelling faster than the speed of light, and so p and q are spacelike separated while m can send a signal and so m and p are timelike separated. We can relate this to the metric as follows. Two events are

- spacelike separated if $ds^2 = -dt^2 + dx^2 > 0$
- timelike separated if $ds^2 = -dt^2 + dx^2 < 0$
- null or lightlike if $ds^2 = -dt^2 + dx^2 = 0$.

The same classification applies for geodesics, i.e. a geodesic γ is spacelike if $g(\dot{\gamma}, \dot{\gamma}) > 0$, timelike if $g(\dot{\gamma}, \dot{\gamma}) < 0$ and it is called a null geodesic if $g(\dot{\gamma}, \dot{\gamma}) = 0$. In the Minkowski solution, a null geodesic is the path that a light ray follows.

Now let's look at the next simplest vacuum solution, the Schwarzschild solution, which depicts the gravitational field outside a spherical object of mass m.

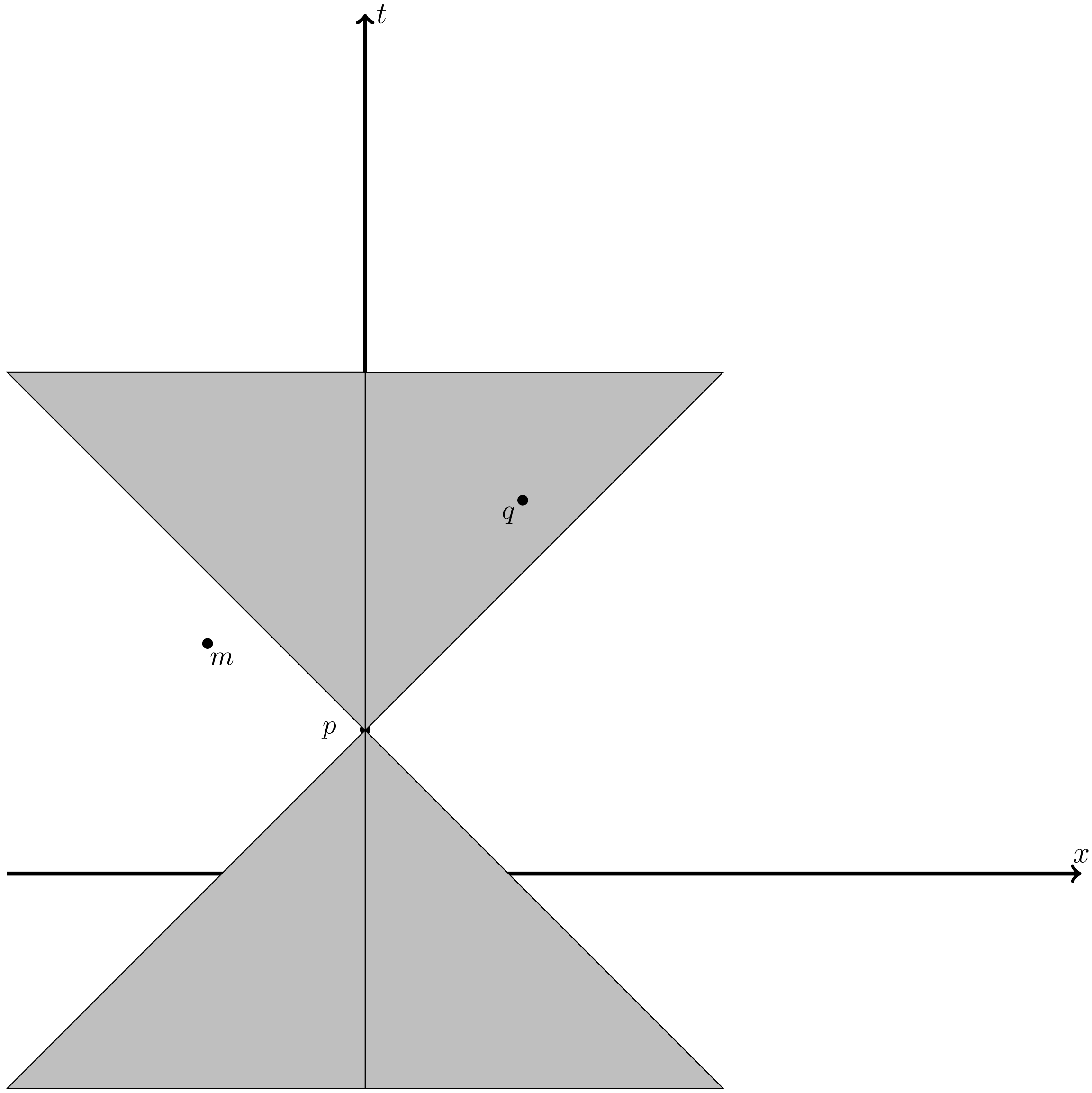


Figure 3: A light cone in Minkowski space at some point p. The shaded region represents the area not causally connected to p.

5 The Schwarzschild solution

The Schwarzschild solution is

$$g = (1 - \frac{2m}{r})dt^2 - (1 - \frac{2m}{r})^{-1}dr^2 - r^2(d\theta^2 + \sin^2(\theta)d\phi^2). \tag{80}$$

We are faced with two questions:

1. What happens at $r = 2m$?
2. Is there anything beyond $t \to \pm\infty$?

To look at these issues, we need transcend the co-ordinates that we have chosen, and look objective objects. The objective objects we will look at are geodesics of the spacetime. We will investigate radial null geodesics, which are the paths light rays follow. The radial requirement implies that

$$\dot{\theta} = \dot{\phi} = 0, \tag{81}$$

where dots represent derivatives with respect to some parameter λ, and so the radial null geodesic requirement implies that

$$(1 - \frac{2m}{r})\dot{t}^2 - (1 - \frac{2m}{r})^{-1}\dot{r}^2 = 0 \tag{82}$$

Note that how we choose to parametrise time and hence the other variables is arbitrary, the other variables need not be parametrised in terms of time, t, and so we use here an arbitrary parameter λ.

Now applying the Euler-Lagrange equation for the t equation we get that

$$(1 - \frac{2m}{r})\dot{t} = k \tag{83}$$

where k is some constant. Substituting this into 82 we get that

$$\dot{r}^2 = k^2 \tag{84}$$

and so

$$r(\lambda) = \pm k\lambda. \tag{85}$$

Now, we can just use r to parametrise t instead of λ and so we consider

$$\tilde{t}(r) = t(\pm k\lambda). \tag{86}$$

Restricting to the positive case, we want an equation that relates $\tilde{t}$ to r and so we begin by finding

$$\frac{d\tilde{t}}{dr} = \frac{d\tilde{t}}{d\lambda}\frac{d\lambda}{dr} = \frac{\dot{\tilde{t}}}{\dot{r}} = \frac{\lambda\dot{t}}{\lambda k} = \frac{r}{r-2m} \tag{87}$$

and integrating gives that

$$\tilde{t}_{+}(r) = r + 2m\ln|r-2m|. \tag{88}$$

These are the outgoing null geodesics. Following the same procedure for the negative case we get that

$$\tilde{t}_{-}(r) = -r - 2m\ln|r-2m|. \tag{89}$$

These are drawn on a spacetime diagram in figure 4.

Now it actually appears that our geodesics actually go around the singularity at $r = 2m$, which is the first sign that the singularity $r = 2m$ is a co-ordinate singularity. It appears that our chart just doesn't show what the geodesics do at $t \to \pm\infty$. In fact, we can change co-ordinates so that in going geodesics have gradient -1, these are known as Eddington-Finklestein co-ordinates and this shown in figure 5.

We can see that the singularity at $r = 2m$ is completely removed, the ingoing geodesics pass by it undisturbed. However, this diagram is still not useful when it comes to determining causal structure as it was for our Minkowski space solution. Further co-ordinate transformations and compactification of the diagram lead to the the Penrose diagram for the Schwarzschild black hole spacetime and is shown in figure 6. This shows the maximally extended Schwarzschild solution.

In the Penrose diagram, radial null geodesics (light rays) have slope of ± 1 and so we can clearly see causal structure. We first note that there are clearly two regions to the spacetime, that are not causally connected, as null geodesics leaving region I falling into the black hole return to region I from the white hole, and similarly for region II. This means that there are two regions that correspond to the region outside of the blackhole $r > 2m$ (to the right of the asymptote in figures 5 and 4). Clearly, this cannot represent anything physical, but we will come back to this later.

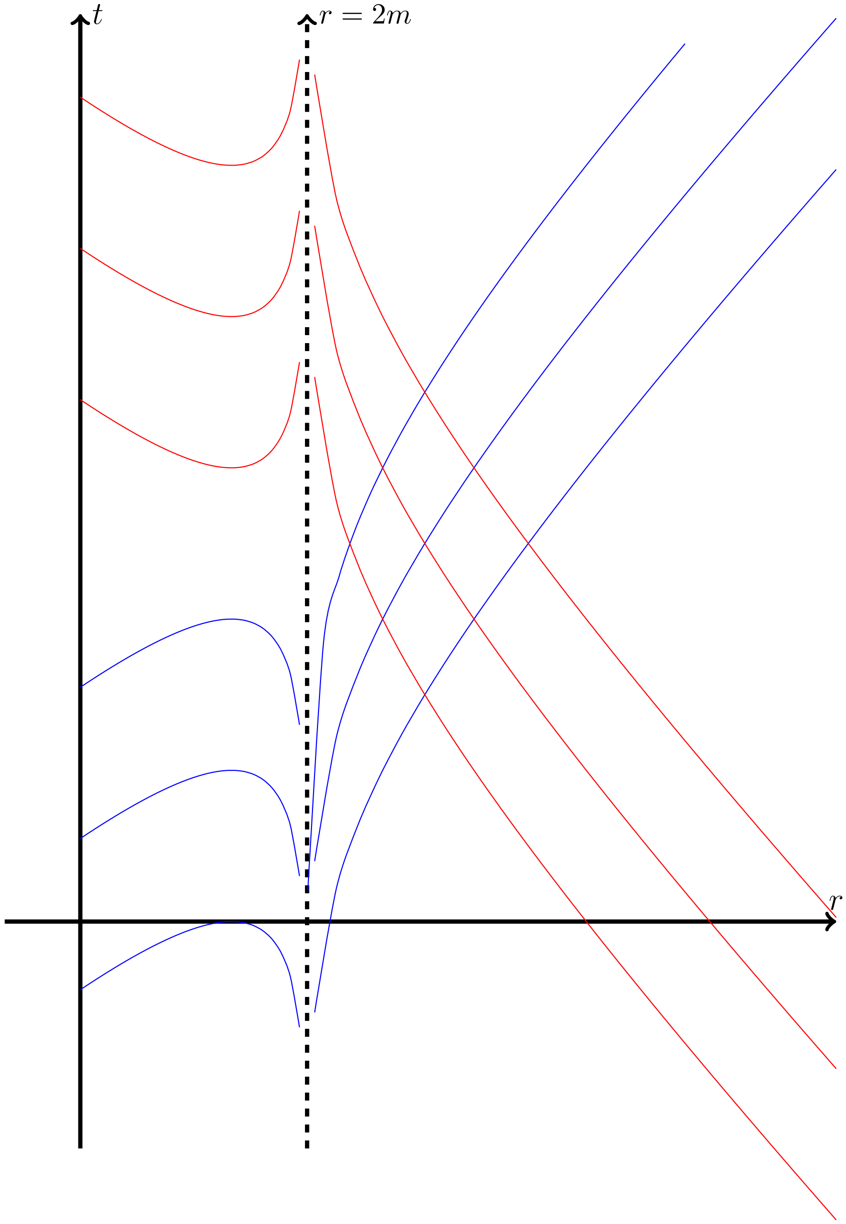


Figure 4: Ingoing (red) and outgoing (blue) null geodesics for the Schwarschild spacetime

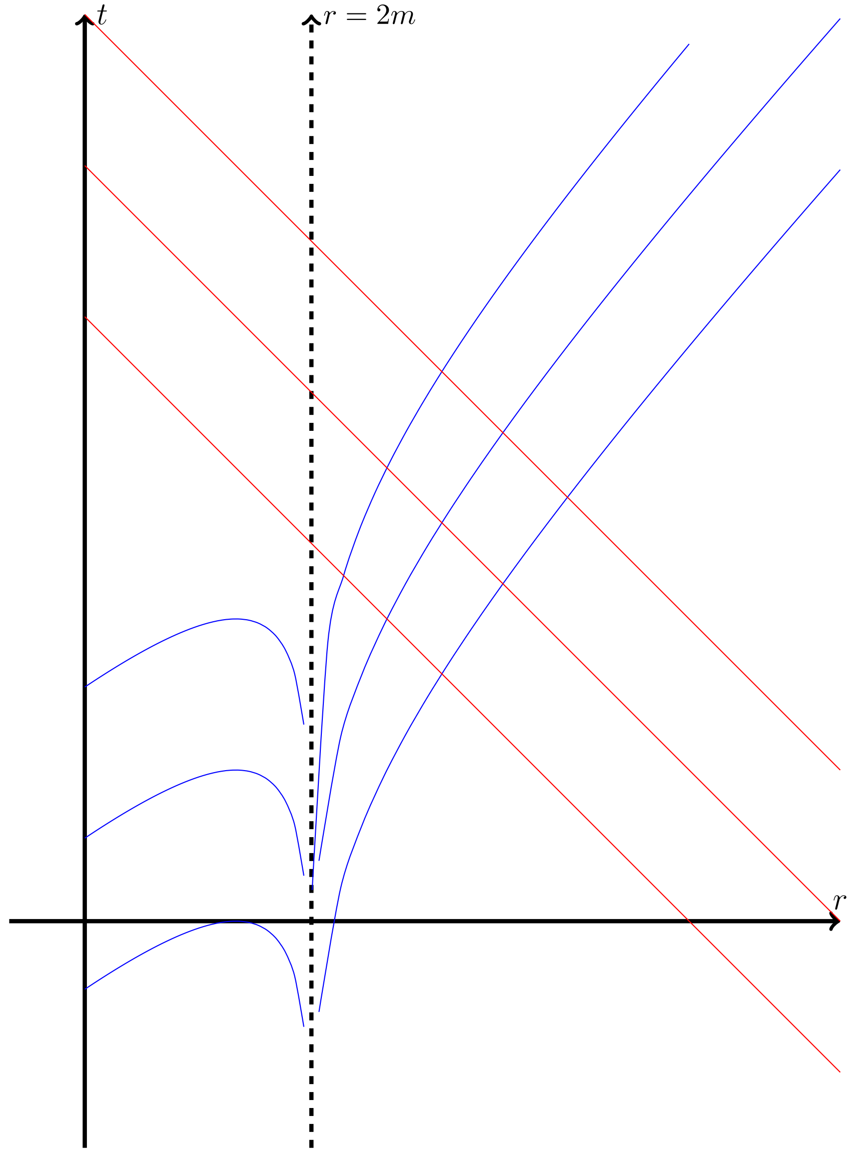


Figure 5: Ingoing (red) and outgoing (blue) null geodesics for the Schwarschild spacetime in Eddington-Finklestein co-ordinates

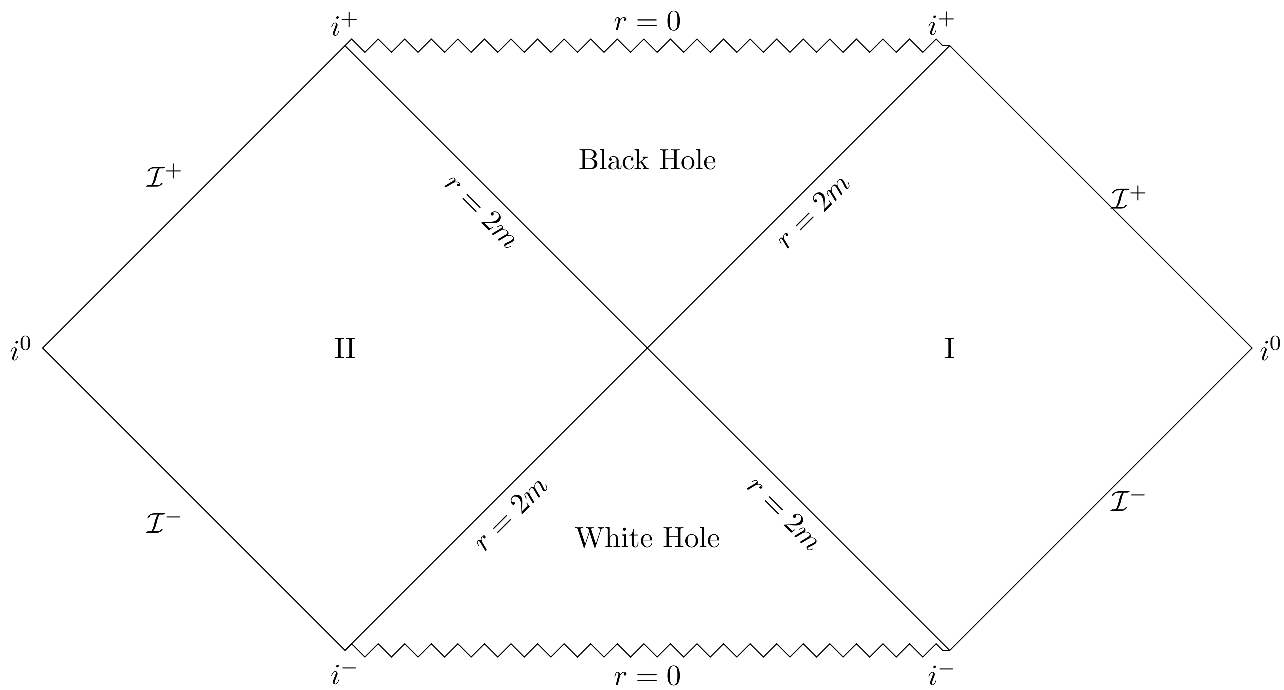


Figure 6: The maximal Schwarzschild solution represented in a Penrose diagram

The next thing that we would like to point out is what we have labelled as $\mathcal{I}^+$ and $\mathcal{I}^-$ which are called future and past null infinity respectively. Formally, $\mathcal{I}^+$ are the limit points of future directed null geodesics, so light rays travelling forward in time will end up somewhere along these boundaries. Thinking of physical applications, the Schwarzschild spacetime only represents the gravitational field around some object such as a star with mass m. So, $\mathcal{I}^+$, would represent far away observers who would receive light from this star. On the other hand, $\mathcal{I}^-$ is where past directed null rays end up, but since time runs forward, we can think of this as being where null geodesics start from so that the path of null geodesics begins at $\mathcal{I}^-$ and ends at $\mathcal{I}^+$.

The rest of the boundary is as follows; i^0 is known as spacelike infinity and $i^\pm$ are known as future and past timelike infinity. Spacelike geodesics start and end at i^0 and timelike geodesics start at i^- and end at i^+.

We can see from the diagram that there is no null geodesic that can leave the black hole and reach future null infinity. Physically, even though observers can observe for all time, this means the observers at $\mathcal{I}^+$ will not be able to observe anything from the black hole region. The boundary at which observers at $\mathcal{I}^+$ stop being able to receive information is $r = 2m$ which is clear from figure 6, this is known as the event horizon. We note that the spacetime does not need a blackhole in order for it to have a Penrose diagram, for example, figure 7 is the Penrose diagram for Minkowski space.

So, just like our Minkowski space solution, we were able to easily represent causal structure for our Schwarzschild solution. But what about the non-physicality of the Schwarzschild solution shown? Why do we even bother with it if it's not a physical situation? The answer is lies in the picture of a physical blackhole spacetime, known as the Oppenheimer-Snyder solution for the gravitational field outside of a collapsing ball of dust [5]. The solution is shown in figure 8 and can be thought of as stellar collapse.

Upon examining 8 we can see the usefulness in studying maximally extended Schwarzschild, the region outside the star is the same as a portion of the Schwarzschild Penrose diagram in 6. In fact, once the star completely collapses, the curved surface of the star reaches $r = 0$, it will all be exactly like a portion of the Schwarzschild case. So although the Schwarzschild solution is a non-physical blackhole, it is very closely related to an actual physical blackhole solution.

6 The Friedmann Equations

In this final section, we will illustrate the use of the Einstein equations in cosmology and why we keep the Λ term in the Einstein equations. As opposed to looking at special solutions, this takes the Einstein equations along with a matter distribution and a metric

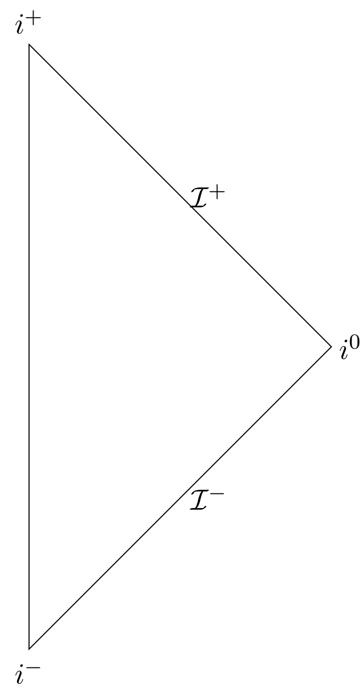


Figure 7: Minkowski space maximally extended

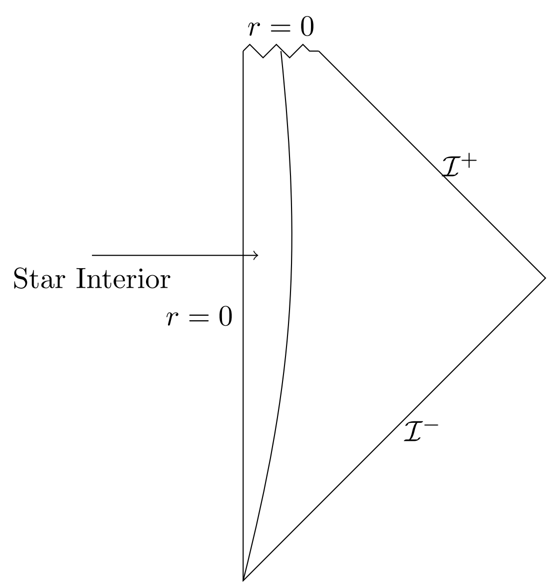


Figure 8: The Oppenheimer-Snyder solution

and then derives a relationship from them. This is another example of one way to view/use the Einstein field equations.

The Robertson Walker Metric

$$g = dt^2 - a^2(t)\Big(\frac{1}{1-\frac{r^2}{K^2}}dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2)\Big), \tag{90}$$

where $a(t)$ is the scale factor and K^{-2} describes the geometry of the universe, whether it is flat $K^{-2} = 0$, open $K^{-2} > 0$ or closed $K^{-2} < 0$, describes the evolution of a homogeneous and isotropic universe.

Since we model the universe as a perfect fluid, the energy momentum tensor that we will use is

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + Pg_{\mu\nu}. \tag{91}$$

Since the metric is diagonal, the only non-zero equations will be of the form

$$R_{ii} - \frac{1}{2}g_{ii}R - g_{ii}\Lambda = 8\pi G T_{ii}. \tag{92}$$

We begin with the t equation which yields,

$$(\frac{\dot{a}}{a})^2 + \frac{1}{Ka^2} = \frac{8\pi G\rho + \Lambda}{3} \tag{93}$$

and this is known as the first independent Friedmann equation. The r, θ and ϕ equations all yield the same equation,

$$\frac{\ddot{a}}{a} + \frac{1}{2}(\frac{\dot{a}}{a})^2 = -4\pi GP + \frac{\Lambda}{2} - \frac{1}{2}\frac{1}{K^2a^2}. \tag{94}$$

Now, multiplying 94 by -2 and then adding it to 93, we get to what is known as the second independent Friedmann equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) + \frac{\Lambda}{3}. \tag{95}$$

Just looking at 95, if there were no cosmological constant, the expansion rate of the universe will be decreasing. However, we know from observations that the universe is expanding

[4] and at an increasing rate of expansion, and so the cosmological constant Λ must be be positive and larger than $\frac{4\pi G}{3}(\rho + 3P)$. This answers more concretely why the cosmological constant is kept in the Einstein equations even though Einstein claims it isn't required.

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