Today we will discuss another important result in functional analysis, the open mapping theorem, which is again a consequence of Baire's lemma. What is an open map I hear you ask? Well, let me tell you...

Definition 1 *A map $T \in L(X,Y)$ is called an open map if $T(U) \subset Y$ is open for every $U \subset X$ open.*

Now that we have this, we can begin with the proof of

Theorem 2 (Open Mapping Theorem) *Let X and Y be Banach spaces and $T \in L(X,Y)$. Then,*

1. *if T is surjective, then T is an open map;*
2. *if T is bijective, then $T^{-1} \in L(Y,X)$.*

I really like this proof, it's quite geometric and very clever. I am following the proof contained in Brezis, "Functional Analysis, Sobolev Spaces and Partial Differential Equations."

Proof: To prove (1) we first show that there exists $r > 0$ so that

$$B_{2r}(0) \subset \overline{T(B_1(0))}. \tag{1}$$

By now Baire's lemma should be jumping out at you! We note that since T is surjective we have that

$$Y = \bigcup_{n \in \mathbb{N}} n\overline{T(B_1(0))},$$

and so applying Baire means there exists a $y \in T(B_1(0))$ and a constant $r > 0$ so that

$$B_{4r}(y) \subset \overline{T(B_1(0))}.$$

We are almost there, we just need to translate the ball to be centered around 0. However, if we translate $B_{4r}(y)$ by $-y$ we won't necessarily end up staying in $\overline{T(B_1(0))}$. So our task now is to figure out what set our ball ends up in after translation.

Since the unit ball in X is symmetrical it follows that $-T^{-1}(y) \in B_1(0)$ and so $-y \in \overline{T(B_1(0))}$. Furthermore, the set $T(B_1(0))$ is convex, again, since $B_1(0)$ is convex. So in fact, after translating, the ball we find is now

$$B_{4r}(0) \subset \overline{T(B_1(0))} + \overline{T(B_1(0))}.$$

We then have (1).

We now would like to show that

$$B_r(0) \subset T(B_1(0)). \tag{2}$$

This is the super clever part. We fix $y \in B_r^Y(0)$, and we look for an $x \in B_1^X(0)$ such that $Tx = y$.

Dividing (1) by 2 we can deduce that for any $0 < \epsilon < \operatorname{diam}(\overline{T(B_1(0))})$ there exists a point $z \in B_{1/2}^X(0)$ whose image achieves this distance ϵ from y. That is, $\|y - Tz\| \leq \epsilon$.

So choosing $\epsilon = r/2$ we can find $x_1 \in B_{1/2}^X(0)$ such that $\|y - Tx_1\| \leq \frac{r}{2}$.

Dividing (1) by 4 and repeating the above step with $y - Tx_1$ and $\epsilon = \frac{r}{4}$ we can see that there exists a $x_2 \in B_{1/4}^X(0)$ such that

$$\|(y - Tx_1) - x_1\| = \|y - T(x_1 + x_2)\| \leq \frac{r}{4}.$$

Continuing this inductively we get a sequence, $(x_n)_{n\in\mathbb{N}} \subset X$ with $\|x_n\| \leq 2^{-n}$. We can then create a new sequence $z_n = x_1 + \cdots + x_n$ with

$$\|y - T(z_n)\| \leq 2^{-n} r.$$

It then follows that z_n is Cauchy (just write out the definition and use geometric series) and so converges to some x with $\|x\| \leq 1$ (triangle inequality with geometric series) and $y = Tx$ (pass the limit inside the previous identity).

Now that we have shown (2) showing that $T(U)$ is open for any open set U is simple! (try it out in the comments)

To show (2) we just need to note that since T is bijective, $(T^{-1})^{-1}(U) = T(U)$ and since T is open, $T(U)$ is open for any open set U. Hence, the pre-image of any open set is open and T^{-1} is continuous. $\square$

In the theme of my blogs, I like to keep them super short. I hope you enjoyed reading it and you found the tricks in this proof super cool. Hopefully we will see some problems coming up which will use this theorem.

Something to think about: Let c_c be the set of comactly supported sequences and consider $T : c_c \to c_c$ where $T(x_n)_{n\in\mathbb{N}} = (\frac{1}{n} x_n)_{n\in\mathbb{N}}$. Check that T is bijective and continuous and consider if T^{-1} is continuous. How does this relate to the open mapping theorem?