A stupid mistake

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So today at lunch a friend of mine posed the following simple problem, given a continuous bijection between two metric spaces, is it's inverse also continuous.

The answer to this question is no, a counter example will soon be given. However, what started with a terribly incorrect proof by me (to be fair I was preoccupied with my food) ended in over an hour of us questioning some very basic fundamental results of continuous linear functionals.

The counter example my friend came up with is the following. Take $X = \mathbb{R}$ and equip it with the discrete metric, that is

$$d_X(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y. \end{cases} \tag{1}$$

Then take $Y = \mathbb{R}$ equipped with the standard metric, $d_Y(x, y) = |x - y|$. Then $T : X \to Y$ where $x \mapsto x$ gives a neat counterexample. Indeed any set in X is open however this is not true in Y and so the inverse cannot be continuous.

However this was not so interesting, or what occupied most of our time. I began by remarking that if we have a continuous linear operator that is a bijection between two Banach spaces then it has to have a continuous inverse (this is the open mapping theorem). To both of us, this example seemed like it violated the open mapping theorem. Then, my friend then exclaimed that our problem is even more fundamental, that we have a continuous linear map that is unbounded. Indeed we went around in circles proving that T is continuous and that T is unbounded until we realised our stupid mistake.

The result that a continuous linear operator is a bounded linear operator (and vice versa) holds in normed vector spaces and the discrete metric is not induced from a norm (just try to prove homogeneity). It turns out, this was the only criteria we didn't check at all, that we were working on metric spaces and not normed vector spaces. Moreover the open mapping theorem I stated above also only holds for complete normed vector spaces (i.e. Banach spaces) and not for complete metric spaces.

This seems like a good point to stop and remind ourselves of some pretty basic facts. We have the following hierarchy:

Inner Product Space $\subset$ Normed Vector Space $\subset$ Metric Space $\subset$ Topological Space.

Moreover, a norm is induced from an inner product iff the norm satisfies the parallelogram identity. A metric is induced from a norm iff it is homogeneous and translation invariant. A topology is metrizable if it is Hausdorff, second countable and regular.