Analysis 3 - PVK

January 2022

Contents

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I: Forecword & Disclaimer

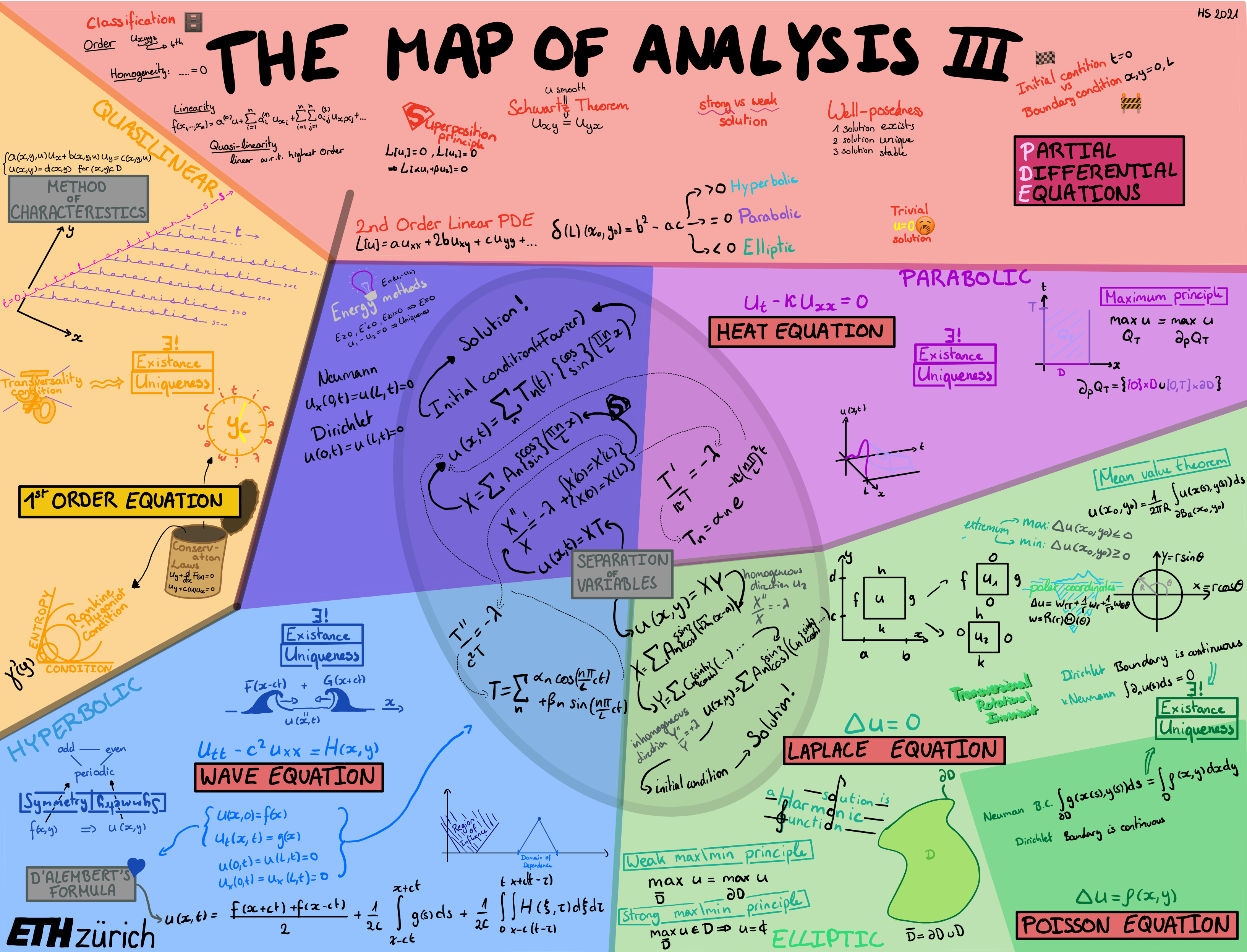
This manuscript is based on the 2021 and 2019 Analysis III lectures of Prof. Iacobelli.

It will serve as the base of the 2022 AMIV PVK.

It was put together by Jean Mégret (megretj) and Anthony Salib (asalib) along with an exercise script.
All the material (including a notability version of this script) will be made available on the AMIV website and on https://n.ethz.ch/~megretj

If you find mistakes or think we should change stuff, please contact us by email.

This manuscript is nowhere near complete with all the lecture content and only targets the (to our eyes) most relevant pieces of theory in order to perform well at the exam.
We do not take any responsability in providing completeness nor correctness in this script.



HS 2021
THE MAP OF ANALYSIS III
Classification
Order
Homogeneity: ... = 0
Linearity
Quasi-linearity
linear w.r.t. highest order
Superposition principle
L[u₁]=0, L[u₂]=0 ⇒ L[αu₁+βu₂]=0
u smooth
Schwartz Theorem
Uxy = Uyx
strong vs weak solution
Well-posedness
1 solution exists
2 solution unique
3 solution stable
Initial condition t=0
vs
Boundary condition x,y=0,L
PARTIAL DIFFERENTIAL EQUATIONS
2nd Order Linear PDE
L[u] = a uxx + 2b uxy + c uyy + ...
δ(L)(x₀,y₀) = b² - ac
> 0 Hyperbolic
= 0 Parabolic
< 0 Elliptic
Trivial u=0 solution
QUASILINEAR
METHOD OF CHARACTERISTICS
t=0 initial condition
characteristics
∃! Existance Uniqueness
Transversality condition ≠ 0
1st ORDER EQUATION
Conservation Laws
Rankine-Hugoniot Condition
ENTROPY CONDITION
Energy methods
Neumann
uₓ(0,t)=u(L,t)=0
Dirichlet
u(0,t)=u(L,t)=0
Solution!
Initial condition + Fourier
SEPARATION OF VARIABLES
u(x,t)=XT
X''/X = -λ
Tₙ = αₙe^{-k(nπ/L)²t}
PARABOLIC
Uₜ - kUxx = 0
HEAT EQUATION
∃! Existance Uniqueness
Maximum principle
max u = max u
Q_T ∂_pQ_T
∂_pQ_T = {0}×D ∪ [0,T]×∂D
Mean value theorem
extremum
max: Δu(x₀,y₀) ≤ 0
min: Δu(x₀,y₀) ≥ 0
polar coordinates
Dirichlet Boundary is continuous
Neumann ∫∂ₙu(s)ds = 0
∃! Existance Uniqueness
Transversal Rotational Invariant
homogeneous direction
inhomogeneous direction
u(x,y)=XY
initial condition → Solution!
Δu = 0
LAPLACE EQUATION
a Harmonic function
solution is
Weak max/min principle
max u = max u
D̄ ∂D
Strong max/min principle
max u ∈ D ⇒ u = c
ELLIPTIC
D̄ = ∂D ∪ D
Neuman B.C.
Dirichlet Boundary is continuous
Δu = ρ(x,y)
POISSON EQUATION
HYPERBOLIC
F(x-ct) + G(x+ct)
odd — even
periodic
Symmetry
∃! Existance Uniqueness
Utt - c²uxx = H(x,y)
WAVE EQUATION
u(x,0)=f(x)
uₜ(x,t)=g(x)
u(0,t)=u(L,t)=0
uₓ(0,t)=uₓ(L,t)=0
Region of Influence
Domain of Dependence
D'ALEMBERT'S FORMULA
ETH zürich

1: Introduction

1.1 Ordinary Differential Equations (ODEs):

ODEs are equations with functions and derivatives of <u>one</u> independent variable, they are the base to solve PDEs, so you should really be familiar on how to solve them.
Just like PDEs, many methods can be used to solve ODEs depending on their form.
So it's important to be able to $\left\{\begin{matrix}\text{distinguish}\\ \text{recognise}\end{matrix}\right\}$ the different types of equations in order to solve them later on.
↳ same for PDEs!
We won't review ODE solving methods here. However, I encourage you to look back at your Analysis course for a refresher. For this course, you will (at the very least) need:

ODEs you should know by heart!

	ODE		ANSATZ	
	$\lambda x(t) = x'(t)$	$\Rightarrow$	$x(t) = Ce^{t}$	$\lambda \in \mathbb{R}$
	$\lambda x(t) = -x''(t)$	$\Rightarrow$	$x(t) = \alpha \sin(\sqrt{\lambda}\,t) + \beta\cos(\sqrt{\lambda}\,t)$	$\lambda \in \mathbb{R}^+$
	$\lambda x(t) = x''(t)$	$\Rightarrow$	$x(t) = \alpha \sinh(\sqrt{\lambda}\,t) + \beta\cosh(\sqrt{\lambda}\,t)$	$\lambda \in \mathbb{R}^+$

w.r.t one variable

a.k.a derivative

1.2 Partial Differential Equations (PDEs):

Order Highest order partial derivative of the function w.r.t. any variable.

u_{xyz^2} : $1+2+1=4$!

Linearity A linear PDE is of the form

$$\boxed{a^{(0)}u+\sum_{i=1}^{n} a_i^{(1)} u_{x_i} + \sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}^{(2)} u_{x_i x_j} + \dots = f(x_1, \dots, x_n)}$$

be aware that the function u and the coefficients $a_{\dots}^{()}$ both depend on the variables $x_1, \dots, x_n$

Quasi-linearity Linear w.r.t the highest order derivative.

Find the highest order derivative and replace it with α (a dummy variable)
Then: is the equation linear w.r.t. α?
If $\exists$ multiple highest deg. terms (eg. u_{yy}, u_{xx}) replace all of them with the same α.
($\Rightarrow$ $0 = u_{xx}u_{yy}$ is not quasilinear)

Homogeneity A linear PDE is homogenious if the right side, i.e. every term that doesn't depend on u, is equal to 0.

Schwartz Theorem Basically: if u is smooth $u_{xy} = u_{yx}$

smooth = C^2, twice continuously differentiable.

↳ very often the case, but not always!

Vector space of solution theorem: (aka superposition principle)

Let $L[u] = f(x)$ be a linear inhomogeneous PDE with solution u_p.
Let $L[u] = 0$ —"— homogeneous —"— solutions u_{h_1} and u_{h_2}

↳ Then $\forall \alpha, \beta \in \mathbb{R}$: $\alpha u_{h_1} + \beta u_{h_2}$ is a solution of $L[u] = 0$
$\alpha u_{h_1} + \beta u_{h_2} + u_p$ —"— $L[u] = f(x)$

During this lecture, we restrict ourselves to functions of two variables:

$$\boxed{u: \mathbb{R}^2 \to \mathbb{R}}$$

2: 1^{st} Order Quasilinear PDEs

2.1 Method of characteristics (M.o.C.)

The method of characteristics will helps us solve some $\underbrace{\text{Cauchy problems}}_{\text{(initial condition + PDE)}}$ of 1^{st} Order quasilinear PDEs. However, in this lecture, we only look at either conservation laws (particular type of quasilinear, we'll see this right after) or linear equations of the form:

1. Order, linear, 2 variables

$$\boxed{a(x,y)\,u_x + b(x,y)\,u_y = c_0(x,y)\,u + c_1(x,y) = c(x,y,u)}$$

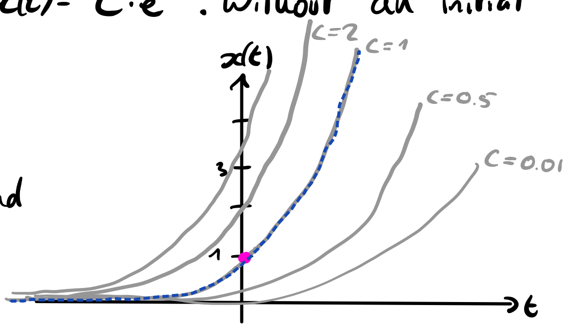
with $\boxed{\underbrace{u(x,y) = d(x,y)}_{\text{initial condition}} \text{ for } x,y \text{ constrained to a domain } D \subset \mathbb{R}^2}$

Only well-posed problems can be solved using the M.o.C.

During the semester we looked at plenty of different initial conditions.

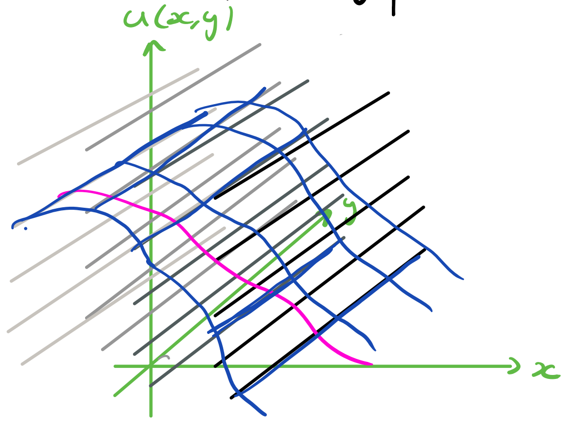
- $u(x,0) = x^2$
- $u(x,y) = 1$ on the unit circle
- ...

The idea behind the M.o.C. is very similar to how we solve ODEs graphically. Take: $2x(t) = x'(t)$, we then know $x(t) = C \cdot e^{2t}$. Without an initial condition $\exists$ ∞-many solutions ($C \in \mathbb{R}$).
But, if we have an initial condition, say $x(0) = 1$,
a unique solution will cross this initial condition and now we have a unique solution!



Now for PDEs, this is slightly more complex, we have an infinite set of solutions in space ($\mathbb{R}^3$) and our initial condition is a path in space (instead of a point in the plane). The ∞-set of solutions will be parametrised by our characteristics. Along with the initial condition, they will knit the unique solution surface $u(x,y)$.

(pardon my poor drawing skills)



However in order to be able to separate the initial condition from the characteristics, we must go through a small procedure.
We can "proove" this prodedure via a graphical interpretation of the problem

Intuition, doesn't need to be understood completely

First, let's rewrite the PDE (dropping out the variables to simplify notation, i.e. $a(x,y)=a$)

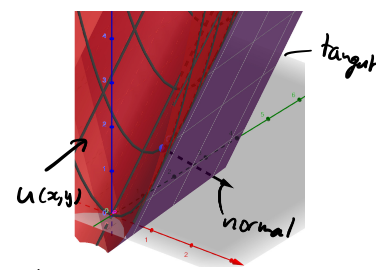
$$a u_x + b u_y - c_0 u - c_1 = 0 \iff \begin{pmatrix} a \\ b \\ c_0 u + c_1 \end{pmatrix} \begin{pmatrix} u_x \\ u_y \\ -1 \end{pmatrix} = 0$$

Does the second form remind you of anything? $\Rightarrow$ its a scalar product between the normal of the surface spanned by u, and another vector!

https://www.geogebra.org/m/kxat7g3h

Example of normal vector $u = x^2 + y$, the normal is $n = \begin{pmatrix} u_x \\ u_y \\ -1 \end{pmatrix} = \begin{pmatrix} 2x \\ 1 \\ -1 \end{pmatrix}$

in the point $Q=(0.5, 1)$, $n = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$



But just, what is this other vector if its scalar product with the normal of the plane $u(x,y)$ is equal to zero?
$\Rightarrow$ It's orthogonal to the normal, so it's a tangent vector of $u(x,y)$!

So $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is somehow related to the tangent of $u(x,y)$ and thus to the derivative of the plane.

In fact if we parametrize x and y trough other variables s and t, $x(s,t), y(s,t)$, we can write:

$$\frac{dx}{dt} = a(x(t), y(t)) \qquad\qquad x(0,s) = x_0(s)$$

$$\frac{dy}{dt} = b(x(t), y(t)) \quad \text{with initial condition} \quad y(0,s) = y_0(s)$$

$$\frac{du}{dt} = c(x(t), y(t)) \qquad\qquad \tilde{u}(0,s) = u_0(s)$$

Which is a set of ODEs, and ODEs we can solve!

<u>So, in a nutshell:</u>

- 1st Order linear PDEs can be seen as a scalar product between 2 vectors.
- This leads us to the intuition that $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ has something to do with the first order derivative of x, y and u.
- This means, we can transform the initial problem into a set of ODEs parametrised by s and t that we can solve for x, y and u.

x,y space
PDE + initial condition $\xrightarrow{x(s,t) \quad y(s,t)}$ s,t space
We choose s, so that it parametrizes the initial condition and "frees" itself from the PDE. So now there's only one Variable: t! $\Rightarrow$ ODE

$\downarrow$ solve ODE

solution $u(x,y)$ $\xleftarrow{s(x,y),\ t(x,y)}$ solution $\tilde{u}(s,t)$

($\rightsquigarrow$ see book chap. 2.3 for a more in depth explanation)

That's enough intuition for now, what you should really be able to do is to solve problems. For this you can follow this procedure:

Possible method for M.o.C.

① Identify components in the equation (a,b,c,d,D)

② Parametrize the initial condition $\Gamma(s) = \begin{pmatrix} x(0,s) \\ y(0,s) \\ \tilde{u}(0,s) \end{pmatrix}$

③ Write down the characteristic equations and solve this set of coupled ODEs to find $x(s,t), y(s,t), u(s,t)$

$x_t = a \qquad x(0,s) = x_0(s)$
$y_t = b \qquad y(0,s) = y_0(s)$
$\tilde{u}_t = c \qquad \tilde{u}(0,s) = \tilde{u}_0(s)$

④ Find inverse mapping for $t \to t(x,y)$ and $s \to s(x,y)$.

⑤ Plug in $\tilde{u}$ to find the final solution $u(x,y)$ and insert solution in problem to check it.

Unfortunately depending on the problem, a step of the method might not work and there might not even be a solution!

Obstacles towards global solution

(i) Solution might blow up in finite time

(ii) Characteristics intersect initial curve more than once.

(iii) Characteristics intersect with each other.

(iv) If vector field (a,b) vanishes at some point.

Hopefully, however there's a way to check whether $\exists!$ (there exists a unique) solution, before starting to solve everything.

Existence and Uniqueness Theorem

Assume $\exists\, s_0 \in \mathbb{R}$ s.t. the transversality condition holds, then $\exists!$ solution u of the Cauchy problem defined in a neighborhood of $(x(0,s_0), y(0,s_0))$.

Note: This means, for a least a little time, there will be a strong solution where the initial condition is transverse to the characteristics. So existance & uniqueness might not hold $\forall t$ (maybe only up until a critical time y_c!)

Transversality Condition

$$J = \det \begin{pmatrix} a(x_0(s), y_0(s), u_0(s)) & b(x_0(s), y_0(s), u_0(s)) \\ \frac{d}{ds} x_0(s) & \frac{d}{ds} y_0(s) \end{pmatrix}$$

$$= \begin{vmatrix} a(0,s) & b(0,s) \\ \frac{d}{ds} x(0,s) & \frac{d}{ds} y(0,s) \end{vmatrix}$$

Recall:

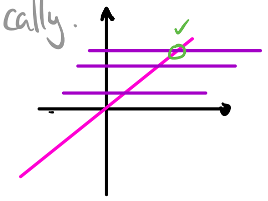
$\det(M) = |M|$

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

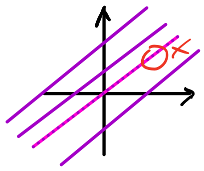
$$= \begin{cases} 0 & \text{for some } s \Rightarrow \text{no solution exists for that } s \\ \neq 0 & \text{for some } s \Rightarrow \text{solution exists for that } s! \end{cases}$$

These are remarks, feel free to read them or not.

Graphically.



initial curve, tangent is $\begin{pmatrix} \frac{d}{ds} x(0,s) \\ \frac{d}{ds} y(0,s) \end{pmatrix}$



characteristics, tangent is $\begin{pmatrix} x_t \\ y_t \end{pmatrix}$

are they transverse? (= not tangential)
if so, the characteristics can propagate information away from the initial curve.
Remember both should "knit" the solution surface

→ Example 1

2.2 Conservation Laws (C.L.)

(e.g. electric charge, people, traffic, ...)

Fancy name for PDEs describing the evolution of conserved quantities.
We use x as a spatial variable and y a temporal variable so $\boxed{y>0}$

General Formulation

We look for $u(x,y): \mathbb{R}\times[0,\infty) \to \mathbb{R}$ such that

both are equivalent! either: $u_y + \frac{d}{dx}F(u) = 0$ or: $u_y + c(u)\,u_x = 0$

named "the flux" $F: \mathbb{R}\to\mathbb{R}$

$c(u) = \frac{d}{du}F(u)$

C.L. often come with an initial $\left\{\begin{matrix}\text{condition}\\ \text{data}\end{matrix}\right\}$ $\boxed{u(x,0) = h(x)}$

$t=0$

Example: $u_y + c\,u_x = 0$ with $c\in\mathbb{R}$ is the transport equation. $c(u)=c$, $F(u)=cu$

$u_y + u\,u_x = 0$ is the Burger equation $c(u)=u$, $F(u)=\frac{1}{2}u^2$

Turns out, these type of problems (i.e. incl. initial data) can be solved thanks to our beloved method of characteristics (since they are 1st Order Quasilinear PDEs)!

To help the study of such equations, we notice the following:

- The characteristic equations are of the form: $\begin{cases} x_t = c(u) & \text{with } x_0(s) = s \\ y_t = 1 & \text{with } y_0(s) = 0 \\ \tilde{u}_t = 0 & \text{with } \tilde{u}_0(s) = h(s) \end{cases}$

- The characteristics are straight lines: $y(s,t) = t$
  (so fixed s, variable t) $x(s,t)$ = linear function of y

- $\boxed{u(x,y) = h(x - c(u(x,y))\,y)}$ is an implicit solution to the problem.

implicit solution is when the solution of u depends on u itself.

- If we look at the transversality condition, we see that:

$$J = \begin{pmatrix} x_t & y_t \\ x_s & y_s \end{pmatrix} = \det \begin{pmatrix} c(u) & 1 \\ 1 & 0 \end{pmatrix} \equiv 1 \neq 0$$

By the existance theorem, these equations always have a local solution! But, and this is when it gets spicy, this solution might only hold up until the critical time y_c.

Critical time

It is defined by:

$$\boxed{y_c = \inf_{s \in \mathbb{R} :\, c(u_0(s))_s < 0} \left\{ \frac{-1}{c'(u_0(s)) \cdot u_0'(s)} \right\}}$$

The idea of this formula is to see when the derivative of the solution blows up (i.e. when there are discontinuities). If there are no discontinuities in u_0, then there's also no y_c.

Infimum

The infimum is the greatest lower bound.
Let $S \subset P$ be a set, a lower bound is any element $a \subset P$ s.t.

$$a \leq x \quad \forall x \in S$$

The $\inf(S) = y$ if $y \geq a$ where a is any lower bound of S.

E.g. • Let $P = \mathbb{N}$, $S = \{5, 7, 10\}$, then $1, 2, 3, 4, 5$ are lower bounds but 5 is the highest lower bound.

- $\inf\limits_{x \in \mathbb{R}} (e^x) = 0$
- $\inf\limits_{x \in \mathbb{R} \setminus (2, -\infty)} (x^2) = 4$

Rankine-Hugoniot Condition

$$\boxed{\gamma_y(y) = \frac{F(u^+) - F(u^-)}{u^+ - u^-}}$$

Solutions γ that satisfy the RH-Condition are called shock-waves.

If we then integrate γ_y w.r.t y we get a border $\gamma(y)$. Then:

$$u(x,y) = \begin{cases} \overset{\text{above/left}}{u^-} & x < \gamma(y) \\ \underset{\text{below/right}}{u^+} & x > \gamma(y) \end{cases}$$

To make sure a border really is a good one, it must satisfy the entropy condition.

Entropy condition

$$\boxed{c(u^+) < \gamma_y < c(u^-)}$$

→ Example 2

→ Example 3

3: 2^{nd} Order Linear PDEs

3.1 Classification

In this lecture you only looked at PDEs of max 2 variables. The general form of 2^{nd} Order Linear PDEs is:

$$\boxed{L[u] = \underbrace{a u_{xx} + 2b u_{xy} + c u_{yy}}_{\substack{\text{2nd order terms} \\ \text{"leading terms"} \\ \text{"principal part"}}} + \underbrace{d u_x + e u_y + f u}_{\text{lower order term}} = g}$$

Note:
a,b,c,d,e,f,g are functions of (x,y)...
... not u!
We are studying linear equations, not quasilinear!

Given a point (x_0, y_0) the discriminant is

$$\boxed{\delta(L)(x_0,y_0) = b^2(x_0,y_0) - a(x_0,y_0)c(x_0,y_0)}$$

The discriminant helps us to define the types of 2nd order PDEs as:

hyperbolic if $\delta(L)(x_0,y_0) > 0$ (e.g. $u_{yy} - u_{xx} = 0$, wave equ)

parabolic if $\delta(L)(x_0,y_0) = 0$ (e.g. $u_y - u_{xx} = 0$, heat eqn)

elliptic if $\delta(L)(x_0,y_0) < 0$ (e.g. $u_{xx} + u_{yy} = 0$, Laplace eqn)

Since this definition depends on the point we choose (x_0,y_0), the PDE is classified <u>only locally</u>, it may vary on different parts of the plane (x,y).

2^{nd} Order Linear PDEs

Wave equation

$\exists$ boundary conditions?

- no → homogeneous?
  - yes → homogeneous d'Alembert
  - no → inhomogeneous d'Alembert OR superposition principle
- yes → can use symmetry to eliminate B.C.?
  - yes → extend problem with suitable $f(x), g(x)$ then
  - no → heat equation

heat equation

Separation of Variables $u = X(x)\,T(t)$

PDE

homogeneous?

- no → Insert $u = \sum T_n(t) X_n(t)$ in PDE and solve for every n of T_n separately
- yes →
  - heat: $T_n(t) = e^{-k\left(\frac{n\pi}{L}\right)^2 t}$
  - wave: $T_n(t) = A_n \cos\left(\frac{n\pi}{L}ct\right) + B_n \sin\left(\frac{n\pi}{L}ct\right)$

Boundary Conditions

homogeneous?

- no → Find particular solution w that solves the inhomogeneity. Solve for $v = u - w$ → Type of B.C.?
- yes → Type of B.C.?

Type of B.C.? 3

- Dirichlet → $X(x) = \sum_{n \geq 1} A \sin\left(\frac{n\pi}{L}x\right)$
- Mixed → Do the whole derivation!
- von Neumann → $X(x) = \sum_{n \geq 0} A \cos\left(\frac{n\pi}{L}x\right)$

Poisson equation

Separation of Variables $u = X(x)\,Y(y)$

homogenous?

- no → Find particular solution, subtract from PDE and B.C.
- yes ⇒ Laplace! → Domain
  - rectangular → If necessary boundary splitting → If needed add harmonic polynomial to fulfill condition for existence → solve in both homogeneous direction separately, add up solutions. Don't forget that you're looking for u! (not for $v = u - w$ or $\tilde{u} = u - p_h$)
  - Circular
    - Ball → full / section
    - Ring → full / section
    - → use standard solutions from summary and plug in boundary conditions to find coefficients!

3.2 The wave equation (hyperbolic)

The homogeneous cauchy problem looks like:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & x \in \mathbb{R}, t > 0 \\ u(x,0) = f(x), & x \in \mathbb{R} \\ u_t(x,0) = g(x), & x \in \mathbb{R} \end{cases}$$

In opposition to the M.o.C., finding a solution to such the Cauchy problem is quite straight forward:

D'Alembert's Formula for homogenous wave equation

$$\boxed{u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds}$$

It's possible to extend this formula to nonhomogeneous problems, however, let's first look at a few properties of the solution of the wave equation.

The solution to the wave equation can be decomposed in a forward and a backward travelling wave, i.e. $u(x,t) = \underbrace{F(x-ct)}_{\substack{\text{Forward} \\ \text{(towards positive } x)}} + \underbrace{G(x+ct)}_{\substack{\text{Backward} \\ \text{(towards negative } x)}}$

A solution is a Generalized Solution of the wave equation if $f(x)$ and $g(x)$ are piecewise continuous functions. This way, u is also piecewise continuous.

We call the characteristics the lines parametrized by $x+ct = \alpha$ and $x-ct = \beta$ with $\alpha, \beta \in \mathbb{R}$. On these lines,

- $u(x,t)$ is constant on these lines.
- singularities propagate along the characteristics.

This can be seen on the following surface plot of a solution.

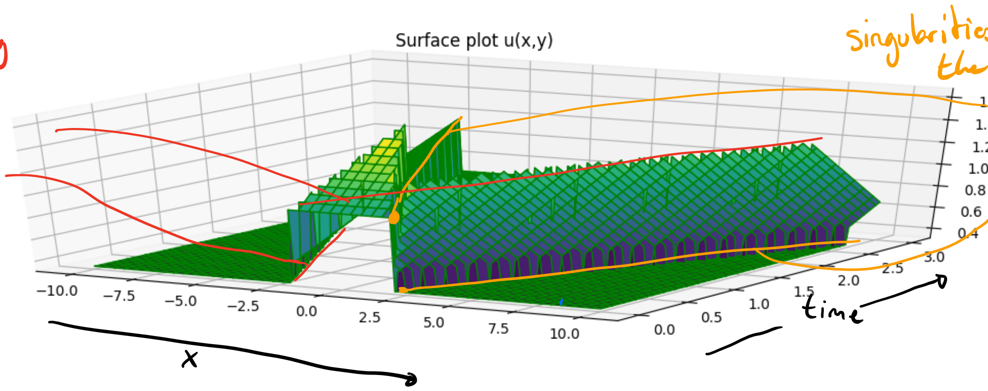
u is constant along the characteristics

$x-3t=-2$

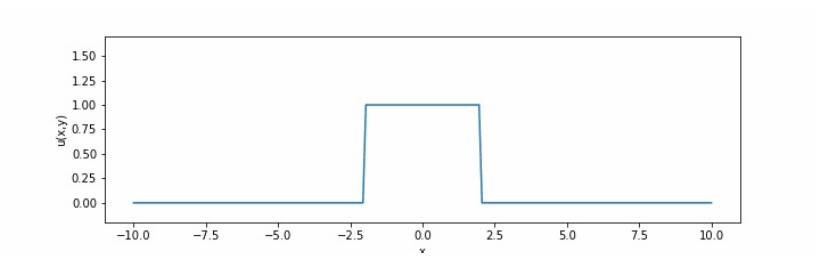
$x+3t=-2$

Singularities follow the characteristics!

$\begin{cases} x+3t=2 \\ x-3t=2 \end{cases}$



Note: The same graph can be visualised as a 2D graph with t representing time. It is more intuitive when we think of time as the second dimension.



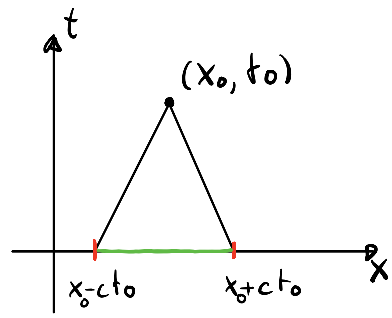
animation available on the website.

Domain of dependence

The solution in (x_0, y_0) depends on

$f(x_0+ct_0)$, $f(x_0-ct_0)$ and

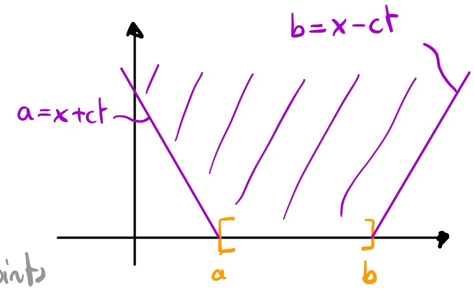
g in the interval $[x_0-ct_0, x_0+ct_0]$



Region of influence

All points satisfying $x-ct \leq b$, $x+ct \geq a$ are dependant on the initial condition on the interval $[a,b]$

So if we change the I.C. inside $[a,b]$, only points in the region of influence will be affected!



Symmetry of wave equation

Let $f(x)$ and $g(x)$ be spacially $\begin{cases} \text{odd} \\ \text{even} \\ \text{periodic} \end{cases}$ functions,

w.r.t. x

then so is $u(x,t)$.

This property can help us solve wave equations with boundary conditions. See exercise 6.3 from this years exercise sheets.

→ Example 4

D'Alembert's Formula for inhomogenious wave equation

Now we're ready for the inhomogeneous wave equation:

The solution to the inhomogeneous Cauchy problem:

$$\begin{cases} u_{tt} - c^2 u_{xx} = F(x,t) & , x \in \mathbb{R}, t \in (0,+\infty) \\ u(x,0) = f(x) & , x \in \mathbb{R} \\ u_t(x,0) = g(x) & , x \in \mathbb{R} \end{cases}$$

exists and is given by:

$$\boxed{u(x,t) = \underbrace{\frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds}_{\text{homogenious problem}} + \underbrace{\frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} F(\xi,\tau)\,d\xi\,d\tau}_{\text{inhomogeneity/particular solution}}}$$

Although it's nice to have a single formula for all problems of this sort, sometimes it's not super convenient to compute the last term. But, in many cases it's possible to shortcut the tedious computation by applying the principle of superposition.

So basically, if we...

1. ... find one particular solution v, ...
2. ... we can define $w = u - v$ and the Cauchy problem will become:

$$\begin{cases} w_{tt} - c^2 w_{xx} = u_{tt} - c^2 u_{xx} - v_{tt} + c^2 v_{xx} = 0 \\ w(x,0) = u(x,0) - v(x,0) = f(x) - v(x,0) \\ w_t(x,0) = u_t(x,0) - v_t(x,0) = g(x) - v_t(x,0) \end{cases}$$

3. This problem is homogeneous and we can solve it with homogenous d'Alembert
4. Finally we find $u = w + v$.

This superposition technique is especially effective if the inhomogeneity consists of the addition of two functions of one variable: $F(x,t) = f_1(x) + f_2(t)$

Uniqueness of the solution of the wave equation. theorem. The solution to

$$\begin{cases} u_{tt} - c^2 u_{xx} = F(x,t) & , x \in \mathbb{R}, t \in (0,+\infty) \\ u(x,0) = f(x) & , x \in \mathbb{R} \\ u_t(x,0) = g(x) & , x \in \mathbb{R} \end{cases}$$

is <u>unique</u>.

(The proof is quite neat and not too difficult so have a look at it in the lecture notes if you have a bit of time!)

→ Example 5

Sometimes, however, using d'Alembert, cannot help us to solve the wave equation with boundary conditions. But we're in luck, introduce: the separation of variables.
Here's the type of problem we will solve (but before that, we will quickly introduce the heat equation)

Wave equation with boundary conditions

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \\ \text{one of } \begin{cases} u(0,t) = u(L,t) = 0 & \text{(Dirichlet)} \\ u_x(0,t) = u_x(L,t) = 0 & \text{(von Neumann)} \\ \text{or mixed} \end{cases} \end{cases}$$

3.3 Heat equation (parabolic)

The general formulation of this homogeneous 2nd order linear PDE is

$$\boxed{u_t - k\, u_{xx} = 0}$$

It is often accompanied by a set of boundary conditions:

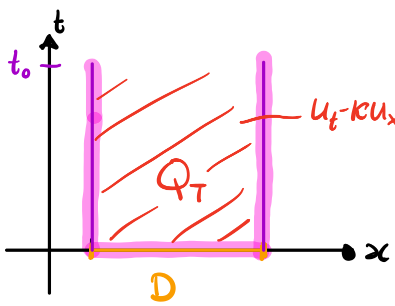
$$\begin{cases} u_t - k\, u_{xx} = 0 & (x,t) \in [0,L] \times [0,\infty) \\ u(x,0) = f(x) & \\ \text{one of } \begin{cases} u(0,t) = u(L,t) = 0 \\ u_x(0,t) = u_x(L,t) = 0 \\ \text{or mixed} \end{cases} & \begin{pmatrix} \text{Dirichlet} \\ \text{von Neumann} \end{pmatrix} \end{cases}$$

Initial data → $u(x,0) = f(x)$

Boundary condition → one of the above

We will see later on how to solve the heat equation. (Spoiler: Separation of variables)

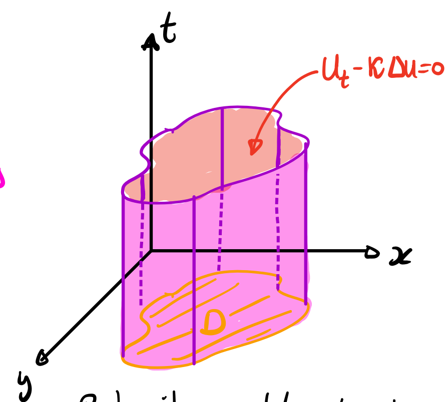
Boundary



Until now, we only solved one dimensional heat equations

The boundary ∂Q_T is defined as

$$\partial_p Q_T = \{ \{0\} \times D \cup [0, t_0] \times \partial D \}$$



But it could also be in higher dimensions:

Uniqueness of Dirichlet problem for the heat equation

$$\begin{cases} u_t - k\Delta u = f & \text{in } Q_T \\ u(0,x) = g & \text{on } D \\ u(t,x) = h & \text{on } [0,T] \times \partial D \end{cases}$$

has a unique solution.

3.4 Laplace and Poisson Equation (elliptic)

Both Laplace and Poisson equations use the Laplace operator

$$\Delta u = \nabla^2 u = \nabla \cdot \nabla u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = \sum_{i=1}^{n} u_{x_i x_i}$$

As we limit our selves to 2 variables in this course, we define the Laplace equation as:

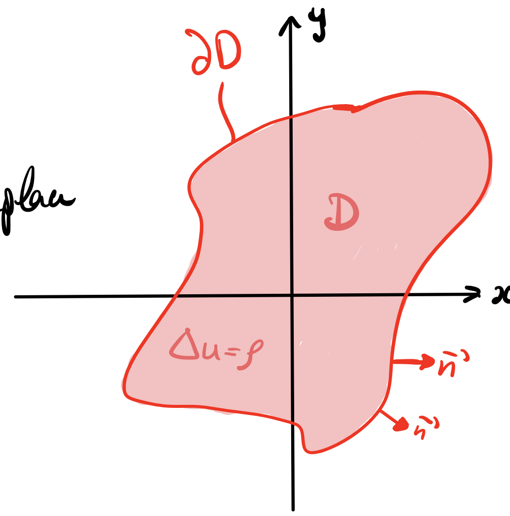
$$\Delta u(x,y) = u_{xx} + u_{yy} = 0$$

Any function that solves the Laplace equation is a Harmonic function.

The Poisson Equation on the other hand is just the inhomogeneous generalization of the Laplace equation:

$$\Delta u(x,y) = \rho(x,y)$$

As for hyperbolic (wave) and parabolic (heat) equations, we can define a Problem. Only now, we've traded our time variable t for another spacial variable y. This means there will be no initial condition but the boundaries will now be in 2D instead of on one axis x.
This boundary is noted as ∂D.



This means, inside a domain $D \in \mathbb{R}^2$, the Poisson/Laplace equation is true:

$$\Delta u = \rho(x,y) \qquad (x,y) \in D$$

$\bar{D} = \partial D \cup D$

and on its border, u must match some boundary condition, either:

Dirichlet $\quad u(x,y) = g(x,y) \qquad (x,y) \in \partial D$

Von Neumann $\quad \partial_n u(x,y) = \vec{n} \cdot \vec{\nabla} u = g(x,y) \qquad (x,y) \in \partial D$

Third kind $\quad u(x,y) + \alpha(x,y)\, \partial_n u(x,y) = g(x,y) \qquad (x,y) \in \partial D$

Separation of variables is the weapon of choice when solving the Laplace equation.

4. Separation of variables

It is a method to solve 2nd Order linear PDEs (heat-, wave-, Laplace-equation) and it accomodates for boundary conditions (spatial-restriction, e.g. $u(x=0,t)=0$)

Contrary to previous solving methods (e.g. d'Alembert formula) it is not a plug & solve method, but requires a good understanding of the whole mathematical derivation.

We will seek non-trivial solutions (i.e. solution $u(x,t) \neq 0$). Indeed $u=0$ is always solution of homogeneous equations, but honestly it's a not very interesting solution 🥱

4.1 For the homogeneous heat and wave equation

The formal derivation is found in the lecture script chap. 5.1. We will simply go over multiple examples to familiarize ourselves with this solving method.

① Identify the Problem: (1.1) PDE: (1.2) Boundary Condition: (1.3) Initial Data:

② Apply separation of variables to PDE $u = X(x)T(t)$ and extract ODEs

(2.1) ODE for X: (2.2) ODE for T:

③ Find general solution for X (2.1). Make a case distinction for λ!

④ Find general solution for T (2.2), using λ from above.

⑤ Formulate general solution for $u(x,t) = XT$

⑥ Use the initial condition to determine the coefficients

⑦ And finally, enjoy and write the full solution down

Inhomogeneous Boundary conditions

If the problem has inhomogeneous boundary conditions (e.g. $u(0,t)=1$, $u(1,t)=1$), find a w that solves this inhomogeneity $(w=1)$ and subtract it from the the PDE: $v=u-w$. Then solve for v and finally, $u=v+w$.

→ Example 6

Although you should really understand the step taken above to come to the solution, there are some "shortcuts" you can take if you recognize the type of Cauchy problem. For the homogeneous wave and heat equations,

Heat: $u_t - c^2 u_{xx} = 0$
boundary conditions
$u(x,0)=f(x)$

Wave: $u_{tt} - c^2 u_{xx} = 0$
boundary conditions
$u(x,0)=f(x)$
$u_t(x,0)=g(x)$

The form of the general solution of T depends on the type of equation:

$T' = -c^2 \lambda T$

$$\Rightarrow \quad T_n = e^{-c^2 \left(\frac{n\pi}{L}\right)^2 t}$$

$T'' = -c^2 \lambda T$

$$\Rightarrow \quad T_n = A_n \cos\left(\frac{n\pi}{L} ct\right) + B_n \sin\left(\frac{n\pi}{L} ct\right)$$

Then, the form of the general solution for X depends on the boundary condition:

$u(0,t)=u(L,t)=0$ Dirichlet

$u_x(0,t)=u_x(L,t)=0$ von Neumann

$u_x(0,t)=u(L,t)=0$ or $u(0,t)=u_x(L,t)$

$$\left.\begin{array}{ll} \text{D.B.C.}: X_n = \sin\left(\frac{n\pi}{L}x\right) & n=1,2,3,\dots \\ \text{v.N.B.C.}: X_n = \cos\left(\frac{n\pi}{L}x\right) & n=0,1,2,3,\dots \end{array}\right\} \lambda_n = \left(\frac{n\pi}{L}\right)^2$$

Combining both we get:

	Heat equation	Wave Equation
D.B.C.	$u(x,t)=\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$	$u(x,t)=\sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left[A_n \cos\left(\frac{n\pi}{L}ct\right)+B_n \sin\left(\frac{n\pi}{L}ct\right)\right]$
N.B.C.	$u(x,t)=\frac{1}{2}B_0+\sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$	$u(x,t)=\frac{A_0+B_0t}{2}+\sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{L}x\right)\left[A_n \cos\left(\frac{cn\pi}{L}t\right)+B_n \sin\left(\frac{cn\pi}{L}t\right)\right]$

The last thing needed are the coefficients A_n and B_n. We find them either by extracting them directly from the initial condition or by computing them thanks to Fourier decomposition.

Fourier expansion

D.B.C.	$A_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$	$A_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$ $B_n=\frac{2}{cn\pi}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx$
N.B.C.	$B_n=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx$	$A_n=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx$ $B_0=\frac{2}{L}\int_0^L g(x)dx,\ B_n=\frac{2}{cn\pi}\int_0^L g(x)\cos\left(\frac{n\pi}{L}x\right)dx$

Please use these shortcuts with caution and convince yourself they make sense by computing them from the beginning at least once before you start using them. There's a high probability that I've written has some mistakes somewhere, so be aware (don't worry, I checked it but still, stay on your guard.)

4.2 Inhomogeneous heat & wave equation

The method for inhomogeneous equations is a little different:

① Identify the Problem: (1.1) PDE: (1.2) Boundary Condition: (1.3) Initial Data:

② Apply separation of variables to homogenous PDE $u = X(x)T(t)$ and extract ODEs

(2.1) ODE for X: ~~(2.2) ODE for T:~~

③ Find general solution for X (2.1). Make a case distinction for λ!

~~④ Find general solution for T (2.2), using λ from above.~~

④ Formulate general solution for $u(x,t) = XT$ with the basis found in ③

(5-6) Insert in inhomogeneous PDE and use the initial condition to determine the coefficients

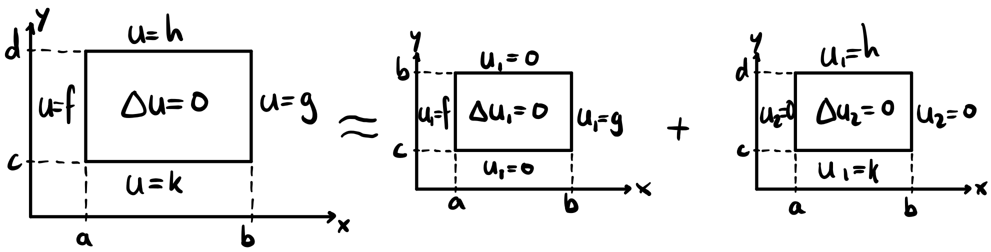
⑦ And finally, enjoy and write the full solution down

→ Example 7

4.3 Laplace equation on rectangular domains

Solving a problem where two opposite sides of the boundary condition is zero is doable. So we use linearity of the Laplace equation to split a problem into two subproblems:

Boundary Splitting



Solve problems for u_1 and u_2 (or $\tilde{u}_1$ and $\tilde{u}_2$), with separation of variables: $u = XY$

- In the homogeneous direction (x for u_2, y of u_1):

  DBC: $X = A_n \sin(\sqrt{\lambda_n}(x-a))$ $\qquad \lambda = \left(\frac{\pi n}{b-a}\right)^2$

  $Y = A_n \sin(\sqrt{\lambda_n}(y-c))$

  NBC: $X = A_n \cos(\sqrt{\lambda_n}(x-a))$

  $Y = A_n \cos(\sqrt{\lambda_n}(y-c))$

- In the other direction

  DBC: $Y = C_n \sinh(\sqrt{\lambda_n}(y-c)) + D_n \sinh(\sqrt{\lambda_n}(y-d))$

  $X = C_n \sinh(\sqrt{\lambda_n}(x-a)) + D_n \sinh(\sqrt{\lambda_n}(x-b))$

  NBC: $Y = C_n \cosh(\sqrt{\lambda_n}(y-c)) + D_n \cosh(\sqrt{\lambda_n}(y-d))$

  $X = C_n \cosh(\sqrt{\lambda_n}(x-a)) + D_n \cosh(\sqrt{\lambda_n}(x-b))$

→ Example 9

Existence of Solution to the Neumann problem

A necessary condition for the existence of a solution to the Neumann problem is:

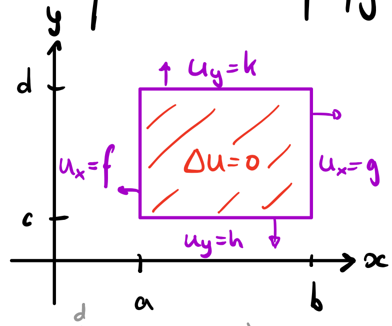
$$\int \partial_n u \qquad \boxed{\underbrace{\int_{\partial D} g(x(s), y(s))\,ds}_{\text{what comes out } \partial D} = \underbrace{\int_D f(x,y)\,dx\,dy}_{\text{what's generated inside } D}}$$

For the Laplace equation, the right hand side is 0.

Existence of solution to the Dirichlet Problem

A necessary condition for the existence of a solution to the Dirichlet problem is continuity of the boundary.

If, in addition, the problem is on a rectangular domain, these equations simplify to:



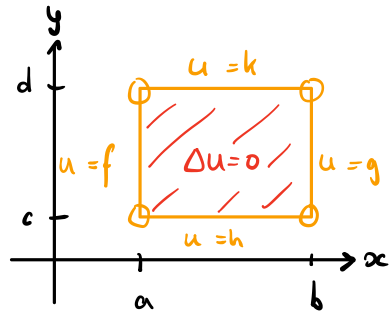
Neumann

$$\oint_{\partial D} \partial_n u(s)\, ds = \int_c^d g\, dy + \int_a^b k\, dx - \int_c^d f\, dy - \int_a^b h\, dx \overset{!}{=} 0$$

"proof": $$= \int_{\partial D} \nabla u\, \vec{n}\, ds = \int_a^b \begin{pmatrix} u_x(x,c) \\ u_y(x,c) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} dx + \int_a^b \begin{pmatrix} u_x(x,d) \\ u_y(x,d) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} dx + \int_c^d \begin{pmatrix} u_x(a,y) \\ u_y(a,y) \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} dy + \int_c^d \begin{pmatrix} u_x(b,y) \\ u_y(b,y) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} dx$$

$$= -\int_a^b u_y(x,c)\,dx + \int_a^b u_y(x,d)\,dx - \int_c^d u_x(a,y)\,dy + \int_c^d u_x(b,y)\,dy = -\int_a^b h\,dx + \int_a^b k\,dx - \int_c^d f\,dy + \int_c^d g\,dy$$

Dirichlet



The boundary must be continuous, in particular,

$$k(a,d) \overset{!}{=} f(a,d) \qquad h(b,c) \overset{!}{=} g(b,c)$$

$$f(a,c) \overset{!}{=} h(a,c) \qquad g(b,d) \overset{!}{=} k(b,d)$$

It is possible that the conditions presented are not met, in that case:

Use linearity and introduce a harmonic polynomial $p_h(x,y) = a_0 + a_1 x + a_2 y + a_3 xy + a_4(x^2 - y^2)$ to u: $\tilde{u} = u + p_H$. Then find the coefficients a_i so that the condition is met. Then solve the problem for $\tilde{u}$. Finally find $u = \tilde{u} - p_h$

→ see Serie 11 2017, exercise 3

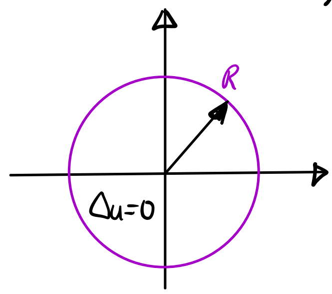
Uniqueness for Dirichlet problem for the Poisson equation

$$\begin{cases} \Delta u = f & \text{in } D \\ u = g & \text{in } \partial D \end{cases} \qquad D \text{ bounded}$$

Then the problem has at most one solution $u \in C^2(D) \cap C(\bar{D})$.

4.4 Laplace equation on circular domain

Instead of being a rectangle, the domain is now a circle (a.k.a circular domain, Ball) of radius a and centred in zero



$$D = B_a = \{0 \leq r \leq a \,,\ \theta \in [0, 2\pi]\}$$

In that case, it's really not that much more complicated. The only extra step is to change the coordinates. Instead of working with x and y, we work with θ and r.
The Laplace equation in polar coordinates is:

$$\Delta u = W_{rr} + \frac{1}{r} W_r + \frac{1}{r^2} W_{\theta\theta} \overset{!}{=} 0 \qquad (u: B_a \to \mathbb{R})$$

where

$$W(r,\theta) = u(r\cos\theta, r\sin\theta) = u(x(r,\theta), y(r,\theta)) \qquad W: B_a \to \mathbb{R}$$

Then, we can use the separation of variables again!

$$W(r,\theta) = R(r)\,\Theta(\theta)$$

Inserting in the Laplace equation, and with the help of Periodicity $\begin{cases} \Theta(0) = \Theta(2\pi) \\ \Theta'(0) = \Theta'(2\pi) \end{cases}$ get a general solution:

$$\boxed{W(r,\theta) = C_0 + \sum_{n=1}^{\infty} r^n \left[A_n \cos(n\theta) + B_n \sin(n\theta)\right]}$$

And as per usual, you can simply insert the boundary conditions to find the coefficients A_n and B_n.

→ Example 8

There are other types of Boundaries, the proof is similar but it's quite unlikely that the other types (2-4) will come to the exam, and if it comes, you can simply insert the boundary condition in the general solution to find the coefficients and solve the problem exactly.

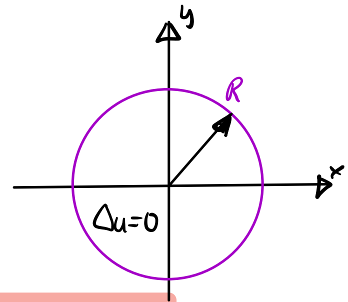
Type 1
Ball

$$\bar{D} = \{0 \leq r \leq R,\ 0 \leq \theta < 2\pi\}$$

Boundary conditions:

$$\left.\begin{aligned} \Theta(0) &= \Theta(2\pi) \\ \Theta'(0) &= \Theta''(2\pi) \end{aligned}\right\} \text{Periodicity}$$

$$\omega(R,\theta) = f(\theta) \ \to \text{given}$$



Solution:

$$\boxed{\omega(r,\theta) = C_0 + \sum_{n=1}^{\infty} r^n \left(A_n \sin(n\theta) + B_n \cos(n\theta)\right)}$$

Insert B.C. and find coefficients.

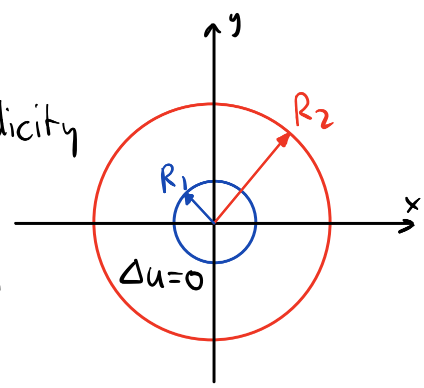
Type 2
Ring

$$\bar{D} = \{R_1 \leq r \leq R_2,\ 0 \leq \theta < 2\pi\}$$

Boundary conditions:

$$\left.\begin{aligned} \Theta(0) &= \Theta(2\pi) \\ \Theta'(0) &= \Theta''(2\pi) \end{aligned}\right\} \text{periodicity}$$

$$\left.\begin{aligned} \omega(R_1,\theta) &= f(\theta) \\ \omega(R_2,\theta) &= g(\theta) \end{aligned}\right\} \text{given}$$



Solution:

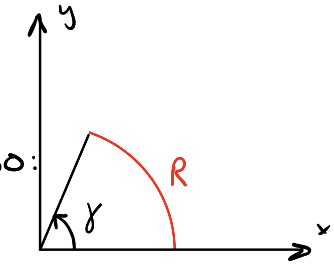
$$\boxed{\omega(r,\theta) = E + F\log(r) + \sum_{n=1}^{\infty} \left\{ r^n \left[A_n \sin(n\theta) + B_n \cos(n\theta)\right] + r^{-n}\left[C_n \sin(n\theta) + D_n \cos(n\theta)\right] \right\}}$$

Type 3
Circle Section

$\bar{D} = \{0 \le r \le R,\ 0 \le \theta \le \gamma\}$

Boundary conditions: $\omega(R,\theta) = h(\theta)$

In this case we only look at N.B.C & D.B.C so:



D.B.C. $\Theta(0) = 0$

$\Theta(\gamma) = 0$

N.B.C. $\Theta'(0) = 0$

$\Theta'(\gamma) = 0$

$$\boxed{\omega(r,\theta) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{\gamma}\theta\right) r^{\frac{n\pi}{\gamma}}}$$

$$\boxed{\omega(r,\theta) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{\gamma}\theta\right) r^{\frac{n\pi}{\gamma}}}$$

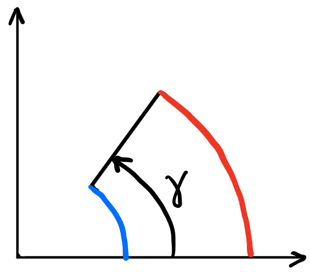
Type 4
Ring Section

$\bar{D} = \{R_1 \le r \le R_2,\ 0 \le \theta \le \gamma\}$

Boundary conditions: $\omega(R_1,\theta) = k(\theta)$

$\omega(R_2,\theta) = h(\theta)$

In this case we look at N.B.C. and DBC:



D.B.C. $\Theta(0) = 0$

$\Theta(\gamma) = 0$

N.B.C. $\Theta'(0) = 0$

$\Theta'(\gamma) = 0$

$$\boxed{\omega(r,\theta) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{\gamma}\theta\right) r^{\frac{n\pi}{\gamma}}}$$

$$\boxed{\omega(r,\theta) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{\gamma}\theta\right) r^{\frac{n\pi}{\gamma}}}$$

5: Maximum principles

Before talking about min/max values of a function you should remember, in 2D:

(x_0, y_0) is an extremum if $\nabla u(x_0,y_0) = \begin{pmatrix} u_x \\ u_y \end{pmatrix}\Big|_{(x_0,y_0)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$

(x_0, y_0) is a maximum if it is an extremum and $\begin{cases} \Delta u(x_0,y_0) \leq 0 \\ \text{OR} \quad u_{xx},\, u_{yy}\big|_{x_0,y_0} \leq 0 \\ \text{OR} \quad D^2 u(x_0,y_0) \leq 0 \end{cases}$

(x_0, y_0) is a minimum if it is an extremum and $\begin{cases} \Delta u(x_0,y_0) \geq 0 \\ \text{OR} \quad u_{xx},\, u_{yy} \geq 0 \\ \text{OR} \quad D^2 u(x_0,y_0) \geq 0 \end{cases}$

This also means, in order for these principles to be valid, they must be C^2.

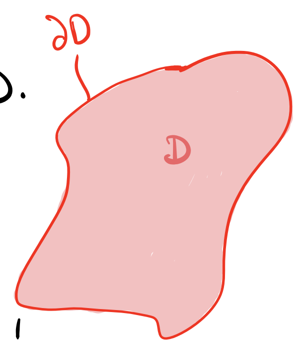
(2 times continuously differentiable)

Weak Maximum /Minimum Principle

Let D be a bounded domain and $u(x,y) \in C^2(D) \cap C(\bar{D})$ a harmonic Function.

$\Rightarrow u(x,y)$ will take its maximum on ∂D.

$$\boxed{\max_{D} u \leq \max_{\bar{D}} u = \max_{\partial D} u}$$



The same can be said for the minimum!

$\bar{D} = \partial D \cup D$

Strong Maximum /Minimum Principle

Let $u(x,y)$ be harmonic in D and u reaches its maximum inside D, then u is constant on all D.

The same can be said for the minimum!

Mean Value Theorem

Let $u(x,y)$ be harmonic in D and let $B_R(x_0,y_0) \subseteq D$ be a Ball of radius R centered in (x_0,y_0). Then:

$$\boxed{u(x_0,y_0) = \frac{1}{2\pi R} \int_{\partial B_R(x_0,y_0)} u(x(s),y(s))\, ds = \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + R\cos\theta, y_0 + R\sin\theta)\, d\theta}$$

Surprisingly, the inverse also holds, so if (*) is satisfied in some domain D then u is harmonic in that domain!

Maximum principle for homogeneous heat equation

Let u solve $u_t = k\Delta u$ in Q_T for some $k>0$.
Assume that D is bounded
Then u achives its maximum (and minimum) on $\partial_p Q_T$

→ Example 10