The Obstacle Problem

Anthony Salib

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Contents

1 Introduction

In this report we develop old and new results related to the obstacle problem in which the right hand side is Hölder continuous. We begin in section 2 by introducing the problem as the minimisation of the Dirichlet energy over a class of H^1 functions which lie above a function called, "the obstacle". We then show how to reduce to the obstacle problem equation $\Delta u = f\chi_{\{u>0\}}$ and prove that solutions u are $C^{1,\alpha}$ for all $\alpha \in (0,1)$ via regularisation.

In section 3 we develop some tools related to sub and super harmonic functions that will be used throughout the report. We then prove the optimal regularity of solutions using the tools of subharmonic and superharmonic functions.

We then move on to the study of the free boundary in section 4. We briefly give some examples of free boundaries and establish the important non-degeneracy property of solutions under the necessary assumption that f is uniformly positive. Section 5 is dedicated to the classification of blow-ups when $f \in C^{0,\alpha}$ which consequently proves the Dichotomy theorem of Caffarelli, up to uniqueness of blow-ups.

We then move on to prove that the free boundary is $C^{1,\delta}$ around regular points in section 6. In this section we restrict to the case $f \equiv 1$ as the case $f \in C^{0,\alpha}$ is treated as a perturbation of the case $f \equiv 1$.

Finally, in the main part of this work, we study the singular set in section 7 where the obstacle problem has right hand side $f \in C^{0,\alpha}$. Specifically, we extend recent results on the fine structure of the singular set to this setting (see [10]).

I would like to thank Federico Franceschini for first of all introducing me to the

problem and helping me throughout the entire way.

2 The Variational Approach to the Obstacle Problem

In this chapter we introduce the obstacle problem in a functional setting and prove the optimal regularity of solutions, following the treatments contained in [7, 14]. The aim of this section is to apply standard tools from functional analysis to the obstacle problem with the highlight being the construction of sufficiently regular solutions via regularisation.

2.1 Existence and Uniqueness

Let $\Omega \subset\subset \mathbb{R}^n$ be an open and smooth domain, $\varphi \in H^1(\Omega)$ and $g \in H^1(\Omega)$. The obstacle problem is to minimise

$$E(w) = \int_\Omega \frac{1}{2}|\nabla w|^2 \tag{2.1}$$

over the set

$$\mathcal{A}_{\phi,g} = \left\{ w \in H^1(\Omega) : w \geq \varphi,\ w - g \in H^1_0(\Omega) \right\}.$$

Note that in order for $\mathcal{A}_{\phi,g}$ to be non-empty, we necessarily have the compatibility condition

$$(\varphi - g)_+ \in H^1_0(\Omega).$$

Furthermore, we note that $\mathcal{A}_{\phi,g}$ is a closed and convex subset of $H^1(\Omega)$ and so it is weakly closed. This, coupled with the fact that the norm on a normed vector space is weakly sequential lower semi-continuous, allows us to apply the usual variational argument to deduce existence and uniqueness of minimisers [18].

Theorem 2.1 (Existence and uniqueness of minimisers)**.** *There exists a unique $v \in \mathcal{A}_{\varphi,g}$ such that*

$$E(v) = \inf_{w \in \mathcal{A}_{\varphi,g}} E(w).$$

Proof. We first take $(v_k)_{k\in\mathbb{N}} \subset \mathcal{A}_{\varphi,g}$ a minimising sequence for E, that is,

$$\lim_{k\to\infty} E(v_k) = \inf_{\tilde{v}\in\mathcal{A}_{\phi,g}} E(\tilde{v}) =: \alpha \geq -\infty.$$

Observe that $(\varphi - g)_+ + g \in \mathcal{A}_{\varphi,g}$ with $\alpha \leq E((\varphi - g)_+ + g) < \infty$. Hence, we have that α is finite and by the convergence of the minimising sequence, for some $\alpha < C < \infty$ there exists an $N \in \mathbb{N}$ such that for any $k \geq N$,

$$E(v_k) \leq C.$$

Thus, we are led to the uniform bound

$$\|\nabla v_k\|_{L^2(\Omega)} \leq C.$$

Then using the Poincaré inequality we obtain

$$\begin{aligned}\|v_k\|_{L^2(\Omega)} &\leq \|v_k - g\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)}\\ &\leq C\|\nabla(v_k - g)\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)}\\ &\leq C\|\nabla v_k\|_{L^2(\Omega)} + C\|g\|_{H^1(\Omega)}.\end{aligned}$$

In particular, $(v_k)_{k\in\mathbb{N}}$ is a bounded sequence in $H^1(\Omega)$.

By the reflexivity of the Hilbert space $H^1(\Omega)$ there exists a subsequence (which we do not relabel) and some $v \in H^1(\Omega)$ such that $v_k \rightharpoonup v$. Since $\mathcal{A}_{\phi,g}$ is weakly closed, $v \in \mathcal{A}_{\phi,g}$.

By weak sequential lower semi-continuity of the L^2 norm, we have that

$$E(v) \leq \liminf_{k\to\infty} E(v_k) = \alpha,$$

and so $E(v) = \inf_{\tilde{v}\in\mathcal{A}_{\varphi,g}} E(\tilde{v})$ as claimed.

Furthermore, since E is strictly convex and $\mathcal{A}_{\varphi,g}$ is convex , v is the unique minimiser for E in $\mathcal{A}_{\varphi,g}$. Indeed if there were two minimisers, v_1 and v_2, we would have

$$E\left(\frac{v_1 + v_2}{2}\right) < \inf_{v\in\mathcal{A}_{\varphi,g}} E,$$

a contradiction. □

The next question we would like to answer is what is the Euler-Lagrange equation satisfied by such a minimiser v.

2.2 Euler-Lagrange Equation

Since we have the additional constraint that any competitor to a minimiser $v \in H^1(\Omega)$ must lie above φ, care must be taken when computing the Euler-Lagrange equation. Specifically, in the region where $u = \varphi$ we can only perturb v by positive functions. Therefore, for any non-negative $\eta \in C_c^\infty(\Omega)$ and $v \in H^1(\Omega)$ a minimiser for E over $\mathcal{A}_{\varphi,g}$, we have by minimality

$$E(v + \varepsilon\eta) \geq E(v), \text{ for all } \varepsilon > 0. \tag{2.2}$$

Expanding and simplifying (2.2) we obtain

$$\int_\Omega \nabla v \cdot \nabla \eta + \frac{\varepsilon}{2}\int_\Omega |\nabla \eta|^2 \geq 0,$$

and then taking $\varepsilon \downarrow 0^+$ we get that

$$\int_\Omega \nabla v \cdot \nabla \eta \geq 0 \text{ for all } \eta \geq 0.$$

Thus, v must satisfy

$$\begin{cases} \Delta v \leq 0 & \text{in } \Omega, \text{ in the sense of distributions,} \\ v \geq \varphi & \text{in } \Omega, \text{ in the sense of distributions,} \\ v = g & \text{on } \partial\Omega, \text{ in the sense of traces.} \end{cases} \tag{2.3}$$

In the perspective we adopt here, this is not the easiest equation to work with. Therefore, to make the analysis simpler, in what follows we will set the obstacle to zero by considering instead $u = v - \varphi$. Moreover we will assume for the remainder of this section higher regularity on the obstacle, that is $\varphi \in C^{1,1}(\Omega)$. Setting $f = -\Delta\varphi \in L^\infty(\Omega)$ and $\tilde{g} = g - \varphi$ we have that u must satisfy

$$\begin{cases} \Delta u \leq f & \text{in } \Omega, \text{ in the sense of distributions,} \\ u \geq 0 & \text{in } \Omega, \text{ in the sense of distributions,} \\ u = \tilde{g} & \text{on } \partial\Omega, \text{ in the sense of traces.} \end{cases} \tag{2.4}$$

A weak solution u to (2.4) can be obtained by minimising

$$E_1(w) = \int_\Omega \frac{1}{2}|\nabla w|^2 + fu$$

over the set

$$\mathcal{A}_{0,\tilde{g}} = \left\{ w \in H^1(\Omega) : w \geq 0,\ w - \tilde{g} \in H^1_0(\Omega) \right\}.$$

We have the existence and uniqueness of solutions to this variational problem as a consequence of Theorem 2.1.

Proposition 2.2. *There exists a unique $u \in \mathcal{A}_{0,\tilde{g}}$ with*

$$E_1(u) = \inf_{w \in \mathcal{A}_{0,\tilde{g}}} E_1(w).$$

Proof. Let $\tilde{v} \in \mathcal{A}_{\varphi,g}$ and define $\tilde{u} = \tilde{v} - \varphi \in \mathcal{A}_{0,\tilde{g}}$. Furthermore we note that any element $\tilde{u} \in \mathcal{A}_{0,\tilde{g}}$ can be obtained in this way, in particular, we have that $\mathcal{A}_{0,\tilde{g}} = -\varphi + \mathcal{A}_{\varphi,g}$. Then substituting $\tilde{v} = \tilde{u} + \varphi$ into E and integrating by parts we attain

$$\begin{aligned} E(\tilde{v}) &= \int_\Omega \frac{1}{2}|\nabla \tilde{v}|^2 \\ &= \int_\Omega \frac{1}{2}|\nabla \tilde{u}|^2 + \int_\Omega \nabla\tilde{u} \cdot \nabla\varphi + \frac{1}{2}\int_\Omega |\nabla\varphi|^2 \\ &= \int_\Omega \frac{1}{2}|\nabla \tilde{u}|^2 + \int_\Omega f\tilde{u} + \frac{1}{2}\int_\Omega |\nabla\varphi|^2 \\ &= E_1(\tilde{u}) + C_\varphi, \end{aligned}$$

where

$$C_\varphi = \frac{1}{2}\int_\Omega |\nabla\varphi|^2$$

is a constant depending only on φ. Consequently, $v \in \mathcal{A}_{\varphi,g}$ will minimise E over $\mathcal{A}_{\varphi,g}$ iff $u = v - \varphi$ minimises E_1 over $\mathcal{A}_{0,\tilde{g}}$. By Theorem 2.1, there exists a unique minimiser of $u \in \mathcal{A}_{0,\tilde{g}}$. □

The following proposition now gives a very useful equivalent variational characterisation of (2.4).

Proposition 2.3. *$u \in \mathcal{A}_{0,\tilde{g}}$ is a minimiser for E_1 over $\mathcal{A}_{0,\tilde{g}}$ if and only if u is a minimiser of*

$$E_2(w) = \int_\Omega \frac{1}{2}|\nabla w|^2 + f w_+$$

over the set

$$\mathcal{A}_{\tilde{g}} = \left\{w \in H^1(\Omega) : w - \tilde{g} \in H^1_0(\Omega)\right\}.$$

Proof. The proposition will follow after showing that any $u \in \mathcal{A}_{\tilde{g}}$ that minimises E_2 is non-negative, since $E_1(w) = E_2(w)$ for any $w \geq 0$.

Suppose that $u \in \mathcal{A}_{\tilde{g}}$ is a minimiser for E_2 and $u_- \not\equiv 0$. Then we can write $u = u_+ - u_-$ with $\nabla u = \nabla u_+ - \nabla u_-$ and

$$\int_\Omega \nabla u_+ \cdot \nabla u_- = 0.$$

Moreover, since $\tilde{g} \geq 0$ on $\partial\Omega$ it follows that $u_+ = \tilde{g}$ on $\partial\Omega$ and so $u_+ \in \mathcal{A}$. It is then immediate that

$$E_2(u) = \int_\Omega \frac{1}{2}|\nabla u_+|^2 + \frac{1}{2}|\nabla u_-|^2 + f u_+ > \int_\Omega \frac{1}{2}|\nabla u_+|^2 + f u_+ = E_2(u_+),$$

that is, the non-negative component of u is a better minimiser for E_2, contradicting that u is the minimiser. Hence we must have that $u_- \equiv 0$. □

We conclude this section with the following important theorem, which will underpin all of our analysis of the free boundary.

Theorem 2.4. *Let u be a minimiser of E_2 over $\mathcal{A}$, then u weakly solves*

$$\Delta u = f\chi_{\{u>0\}} \text{ in } \Omega. \tag{2.5}$$

Before proving Theorem 2.4, we state one very important Corollary. Since $f\chi_{\{u>0\}} \in L^\infty$ it follows that $u \in C^{1,\alpha}(\Omega)$, for any $\alpha \in (0,1)$, see for instance [7, Proposition 2.30]. In the next subsection we prove this more directly by a regularisation process.

The proof of Theorem 2.4 will require the use of the Calderòn-Zygmund theorem [14, Theorem 1.1], which we state here without proof.

Theorem 2.5 (Calderòn-Zygmund)**.** *Let* $u \in L^1(\Omega)$*,* $f \in L^p(\Omega)$*,* $1 < p < \infty$ *such that* $\Delta u = f$ *in the sense of distributions. Then* $u \in W^{2,p}_{loc}(\Omega)$ *and for any* $K \subset\subset \Omega$ *there holds*

$$\|u\|_{W^{2,p}_{loc}(K)} \leq C(n, K, \Omega, p)\left(\|u\|_{L^1(\Omega)} + \|f\|_{L^p(\Omega)}\right).$$

Proof of Theorem 2.4. For any $\eta \in C_c^\infty(\Omega)$, and any $\varepsilon > 0$, we have

$$E_2(u + \varepsilon\eta) \geq E_2(u),$$

by the minimality of u for E_2. Re-arranging and dividing by ε we obtain

$$\int_\Omega \nabla u \cdot \nabla \eta + \int_\Omega f\left(\frac{(u+\varepsilon\eta)_+ - u_+}{\varepsilon}\right) \geq 0. \tag{2.6}$$

Observe that if $u = 0$ then

$$\frac{(u+\varepsilon\eta)_+ - u_+}{\varepsilon} = \begin{cases} 0 & \text{in } \{\eta < 0\} \\ \eta & \text{in } \{\eta \geq 0\} \end{cases} = \eta_+,$$

and consequently we obtain that

$$\lim_{\varepsilon \to 0} \frac{(u+\varepsilon\eta)_+ - u_+}{\varepsilon} = \begin{cases} \eta & \text{in } \{u > 0\} \\ \eta_+ & \text{in } \{u = 0\}. \end{cases}$$

Now passing to the limit $\varepsilon \to 0$ in (2.6) yields

$$\int_\Omega \nabla u \cdot \nabla \eta + \int_\Omega f\eta\chi_{\{u>0\}} + \int_\Omega f\eta_+\chi_{\{u=0\}} \geq 0. \tag{2.7}$$

Now testing with $\eta \geq 0$ in (2.7) gives

$$\int_\Omega \nabla u \cdot \nabla \eta + \int_\Omega f\eta \geq 0 \text{ for all } \eta \geq 0,$$

which is equivalent to

$$\Delta u \leq f \tag{2.8}$$

in the sense of distributions. Similarly, testing with $\eta \leq 0$ in (2.7) yields

$$\int_\Omega \nabla u \cdot \nabla \eta + \int_\Omega f\eta\chi_{\{u>0\}} \geq 0 \text{ for all } \eta \leq 0,$$

which is equivalent to

$$\Delta u \geq f\chi_{\{u>0\}} \tag{2.9}$$

in the sense of distributions. Combining (2.9) and (2.8) we arrive at

$$f\chi_{\{u>0\}} \leq \Delta u \leq f \text{ in } \Omega.$$

and in particular, $\Delta u = f$ in $\{u > 0\}$. Since $f \in L^\infty(\Omega)$ we automatically obtain that $\Delta u \in L^\infty_{loc}(\Omega) \hookrightarrow L^2_{loc}(\Omega)$. Applying Theorem 2.5 with $p = 2$ we have that $u \in W^{2,2}_{loc}(\Omega)$ and so on the level set $\{u = 0\}$ we have that $D^2 u = 0$ almost everywhere, and in particular, $\Delta u = 0$ almost everywhere in $\{u = 0\}$. It then follows that

$$\Delta u = f\chi_{\{u>0\}} \text{ a.e. in } \Omega$$

in the sense of distributions as desired. □

Theorem 2.5 will actually give us that $u \in C^{1,\alpha}$ for any $\alpha \in (0,1)$ by a regularisation process which is the content of the following subsection.

2.3 $C^{1,\alpha}$ regularity

In this section we will build a solution u to (2.5) as the limit of solutions to a sequence of regularised problems. In this way we will see that any solution is of class $C^{1,\alpha}$. Specifically we will prove

Theorem 2.6. *Let u be the unique solution of (2.5). Then $u \in C^{1,\alpha}$ for any $\alpha \in (0,1)$.*

Proof. First define a smooth approximation of $\chi_{\{t>0\}}$ as $\{F_\varepsilon\}_{\varepsilon>0}$ where $F_\varepsilon \in C^1$ with $0 \le F_\varepsilon \le 1$, $F_\varepsilon = 0$ when $t \le -\varepsilon$ and $F_\varepsilon(t) = 1$ when $t \ge \varepsilon$. Now consider the family of solutions $\{u_\varepsilon\}_{\varepsilon>0}$ to the regularised problems

$$\begin{cases} \Delta u_\varepsilon = f(x)F_\varepsilon(u_\varepsilon) & \text{in } \Omega \\ u_\varepsilon = \tilde{g} & \text{on } \partial\Omega. \end{cases} \tag{2.10}$$

Moreover u_ε is a solution to (2.10) iff u_ε minimises

$$\tilde{E}_\varepsilon(v) = \int_\Omega \frac{1}{2}|\nabla v|^2 + f\Phi_\varepsilon(v)$$

over $\mathcal{A}_{\tilde{g}}$, where $\Phi_\varepsilon(t) = \int_{-\infty}^t F_\varepsilon(s)ds$ is the primitive of $F_\varepsilon(t)$.

Claim: $(u_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $H^1(\Omega)$.

Proof. Since $\tilde{g} \in \mathcal{A}_{\tilde{g}}$ we have

$$\frac{1}{2}\|\nabla u_\varepsilon\|^2_{L^2(\Omega)} \le \tilde{E}_\varepsilon(u_\varepsilon) \le \tilde{E}_\varepsilon(g) \le C,$$

and applying the Poincaré inequality we obtain that for all $\varepsilon > 0$

$$\begin{aligned} \|u_\varepsilon\|_{L^2(\Omega)} + \|\nabla u_\varepsilon\|_{L^2(\Omega)} &\le \|u_\varepsilon - g\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)} + \|\nabla u_\varepsilon\|_{L^2(\Omega)} \\ &\le C\|\nabla u_\varepsilon - \nabla g\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)} + \|\nabla u_\varepsilon\|_{L^2(\Omega)} \\ &\le C. \end{aligned}$$

□

As a consequence of the claim we have that there exists a subsequence $\varepsilon_k \downarrow 0$ and some $u \in H^1(\Omega)$ such that $u_{\varepsilon_k} \rightharpoonup u$ in $H^1(\Omega)$, and hence $u_{\varepsilon_k} \to u$ in $L^2(\Omega)$. Moreover, $u_{\varepsilon_k} - g \in H_0^1(\Omega)$ for all $k \in \mathbb{N}$ and so $u - g \in H_0^1(\Omega)$ and we conclude that $u \in \mathcal{A}_{\tilde{g}}$.

By the Calderón-Zygmund estimate we now obtain for any $K \subset\subset \Omega$ the uniform bounds

$$\|u_{\varepsilon_k}\|_{W^{2,p}(K)} \leq C\left(\|u_{\varepsilon_k}\|_{L^2(\Omega)} + \|fF_{\varepsilon_k}(u_{\varepsilon_k})\|_{L^\infty(\Omega)}\right),$$

for all $1 < p < \infty$. So by weak compactness in $W^{2,p}$ we have (up to extracting a further subsequence) that $u_{\varepsilon_k} \rightharpoonup u$ in $W^{2,p}_{loc}(\Omega)$ so that $u \in W^{2,p}_{loc}(\Omega)$.

Taking $p > n$ we have by the Sobolev Embedding theorem that $u \in C^{1,\alpha}_{loc}(\Omega)$ for $\alpha = 1 - \frac{n}{p}$. Since p can be as large as we like, we have in particular that $\alpha \in (0,1)$.

To conclude that u is a minimiser for E_2 over $\mathcal{A}_{\tilde{g}}$ we note that by the lower semicontinuity of the L^2 norm with respect to the weak convergence we have for any $v \in \mathcal{A}_{\tilde{g}}$

$$E_2(u) \leq \liminf_{\varepsilon \to 0} \tilde{E}_\varepsilon(u_\varepsilon) \leq \liminf_{\varepsilon \to 0} \tilde{E}_\varepsilon(v) = E_2(v).$$

The fact that u solves (2.5) follows from Theorem 2.4. $\square$

The final question we would like to answer regarding solutions u to (2.5) is what is their optimal regularity. Clearly, Δu will be discontinuous across $\partial\{u > 0\}$ and so u will not be C^2. In the following section we will show that $u \in C^{1,1}$ is optimal.

2.4 Optimal Regularity

From here we will localise to the unit ball, that is, we will consider solutions to

$$\begin{cases} \Delta u = f\chi_{\{u>0\}} & \text{in } B_1 \\ u \geq 0 & \text{in} B_1, \end{cases} \tag{2.11}$$

where again we are assuming $f \in C^{0,\alpha}(B_1)$.

Theorem 2.7. *Let u be a solution to (2.11), then $u \in C^{1,1}(B_{1/2})$ with*

$$\|u\|_{C^{1,1}(B_{1/2})} \leq C\left(\|u\|_{L^\infty(B_1)} + \|f\|_{C^{0,\alpha}(B_1)}\right).$$

Since $f \in C^{0,\alpha}(B_1)$, u is smooth in the region $\{u > 0\}$, and so away from the free boundary u is smooth (u is trivially smooth in $\{u = 0\}$). Consequently, we only need to study the regularity of u close to free boundary points. As a first step, we show u separates at most quadratically from the free boundary $\Gamma(u) = \partial\{u > 0\}$.

Lemma 2.8. *Let u be a solution to (2.5) and $x_0 \in \overline{B_{1/2}} \cap \{u = 0\}$. Then for any $r \in (0, \frac{1}{4})$ there holds*

$$0 \le \sup_{B_r(x_0)} u \le Cr^2,$$

with $C = C(n, \|f\|_{L^\infty})$.

Proof. We first split $u = v + w$ where $\Delta v = 0$ in $B_{2r}(x_0)$ with $v = u$ on $\partial B_{2r}(x_0)$ while $\Delta w = f\chi_{\{u>0\}}$ in $B_{2r}(x_0)$ and $w = 0$ on $\partial B_{2r}(x_0)$. By the minimum principle $v \ge 0$ in $B_{2r}(x_0)$ and so by Harnack's inequality

$$\begin{aligned}\sup_{B_r(x_0)} u &= \sup_{B_r(x_0)} v + \sup_{B_r(x_0)} w \\ &\le C(n) \inf_{B_{2r}(x_0)} v + \sup_{B_r(x_0)} w \\ &\le C(n) \inf_{B_{2r}(x_0)} u + \sup_{B_r(x_0)} w.\end{aligned}$$

To estimate w we use the comparison principle with the barrier

$$b(x) = \|f\|_{L^\infty} \frac{4r^2 - |x - x_0|^2}{2n}.$$

Since $\Delta(b - w) \le 0$ and $b - w = 0$ on $\partial B_{2r}(x_0)$ we have $b - w \ge 0$ in $B_{2r}(x_0)$. Repeating this with $-w$ we have that $|w| \le b(x)$. Clearly $0 \le b(x) \le \frac{2r^2}{n}\|f\|_{L^\infty}$ in $B_{2r}(x_0)$. Putting all this together we have that

$$\sup_{B_r(x_0)} u \le C(n) \left(\inf_{B_{2r}(x_0)} u + C(n) r^2 \|f\|_{L^\infty(B_{2r}(x_0))} \right).$$

However, $u(x_0) = 0$ and $\|f\|_{L^\infty_{B_{2r}(x_0)}} \le C$ and so we achieve

$$\sup_{B_r(x_0)} u \le Cr^2.$$

□

Now that we have Lemma 2.8 Theorem 2.7 follows by Schauder estimates for the Laplacian.

Proof of Theorem 2.7. Choose any $x_1 \in \{u > 0\} \cap B_{1/2}$ and let $x_0 \in \Gamma(u)$ be the closest free boundary point to x_1. Then define $r = |x_0 - x_1|$ and note $\Delta u(rx) = r^2 f(rx)$ in $B_1(x_1)$. By Schauder estimates in $B_1(x_1)$ applied to $u(rx)$ we obtain

$$\left\|D^2 u\right\|_{L^\infty(B_{r/2})} \le C \left(\frac{1}{r^2} \|u\|_{L^\infty(B_r)} + \|f\|_{C^{0,\alpha}(B_r(x_1))} \right). \tag{2.12}$$

Now by Lemma 2.8 and (2.12) we obtain that

$$\left\|D^2 u\right\|_{L^\infty(B_{r/2})} \le C\left(n, \|f\|_{C^{0,\alpha}}\right),$$

and so $|D^2 u(x_1)| \le C$. Since this holds for any $x_1 \in \{u > 0\} \cap B_{1/2}$ it follows that $u \in C^{1,1}(B_{1/2})$. □

3 Optimal regularity through mean value formulas

We recall that in the previous section the first Euler-Lagrange equation obtained was

$$\Delta v \leq 0.$$

This is actually the property that v is super harmonic which can be exploited to prove the optimal regularity of solutions. The perspective taken here is actually simpler than that in section 2 as we will not require elliptic regularity theory. In fact, we will only need the theory of harmonic, sub/superharmonic functions which we will develop in subsection 3.1. We will also state several useful results that will be used in subsequent sections. Then using this theory we will prove the optimal regularity following the treatment contained in [8].

3.1 Harmonic Functions

Throughout this section we will let $\Omega \subset \mathbb{R}^n$ be bounded with C^1 boundary.

We introduce the notions of weakly sub-harmonic and weakly super-harmonic functions.

Definition 3.1. *A function $u \in H^1_{loc}(\Omega)$ is weakly super-harmonic if for any $\eta \in C_c^\infty(\Omega)$ with $\eta \geq 0$*

$$\int_\Omega \nabla u \cdot \nabla \eta \geq 0. \tag{3.1}$$

Analogously, $u \in H^1_{loc}(\Omega)$ is weakly sub-harmonic if for any $\eta \in C_c^\infty(\Omega)$ with $\eta \geq 0$

$$\int_\Omega \nabla u \cdot \nabla \eta \leq 0. \tag{3.2}$$

We now give an important characterisation of sub/super-harmonic functions.

Theorem 3.2. *A function $u \in H^1_{loc}(\Omega)$ is weakly super-harmonic (respectively weakly sub-harmonic) iff for any $x \in \Omega$ the map*

$$r \mapsto \fint_{B_r(x)} u$$

is non-increasing (respectively non-decreasing) for $r \in (0, dist(x, \partial\Omega))$.

We now have the following very useful results concerning weakly sub/super-harmonic functions.

Proposition 3.3. *A weakly super-harmonic function that is bounded from above is lower semi continuous (up to changing u in a set of measure 0).*

Proof. We find a lower semi-continuous representative for u. For any $x_0 \in \Omega$ we define

$$\tilde{u}(x_0) = \lim_{r \to 0} \fint_{B_r(x_0)} u(x)dx.$$

This limit is well defined since u is weakly super-harmonic and so the integral is non-decreasing as $r \downarrow 0$. Furthermore, by the monotonicity we have that

$$\tilde{u}(x_0) = \sup_{0<r<\operatorname{dist}(x_0,\partial\Omega)} \fint_{B_r(x_0)} u(x)dx. \tag{3.3}$$

Now to see that $\tilde{u}$ is lower semi-continuous we take a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \to x_0$ as $n \to \infty$ and observe that by dominated convergence and (3.3) we have that

$$\begin{aligned}
\fint_{B_r(x_0)} u dx &= \frac{1}{|B_r|} \int_\Omega u \chi_{B_r(x_0)} dx \\
&= \frac{1}{|B_r|} \lim_{n\to\infty} \int_\Omega u \chi_{B_r(x_n)} dx \\
&\leq \liminf_{n\to\infty} \tilde{u}(x_n).
\end{aligned}$$

Taking the limit as $r \downarrow 0$ then yields the lower semi-continuity. By the Lebesgue differentiation theorem, $\tilde{u}(x_0) = u(x_0)$ for almost every $x_0 \in \Omega$ and so $\tilde{u}$ is a lower semi-continuous representative for u. □

Proposition 3.4. *Suppose $(u_n)_{n\in\mathbb{N}}$ is a uniformly bounded sequence of weakly super-harmonic functions that converge pointwise to u. Then u is weakly super-harmonic.*

Proof. Given $x_0 \in \Omega$ and $0 < r_1 < r_2 < \operatorname{dist}(x_0, \partial\Omega)$ we have for any $n \in \mathbb{N}$ that

$$\fint_{B_{r_1}(x_0)} u_n \geq \fint_{B_{r_2}(x_0)} u_n. \tag{3.4}$$

Since the u_n are uniformly bounded we can pass to the limit in (3.4) using dominated convergence to obtain that

$$\fint_{B_{r_1}(x_0)} u \geq \fint_{B_{r_2}(x_0)} u.$$

□

3.2 Euler-Lagrange Equation

Recall from subsection 2.2 that we derived the Euler-Lagrange equation for minimisers v of E over $\mathcal{A}_{\varphi,g}$ (defined in subsection 2.1) as

$$\begin{cases}
\Delta v \leq 0 & \text{in } \Omega, \\
v \geq \varphi & \text{in } \Omega, \\
v = g & \text{on } \partial\Omega.
\end{cases}$$

In this section we will again assume that $\varphi \in C^{1,1}(\Omega)$.

Now Proposition 3.3 tells us that v is in fact lower semi-continuous, which allows us to improve the Euler-Lagrange equation in the region where $v > \varphi$. Indeed, (at least formally), when $v > \varphi$ we would be allowed to perturb v with negative test functions and still remain above φ in the variational argument of subsection 2.2. However, in order to carry this out rigorously, we need to know that $\{v > \varphi\}$ is an open set, which is the content of the following lemma.

Lemma 3.5. *Let v be the minimiser of E over $\mathcal{A}_{\varphi,g}$, then v is lower semi-continuous and consequently $\{v > \varphi\}$ is an open set.*

Proof. Since v is the minimiser of the Dirichlet energy it is necessarily bounded from above, or else we could truncate it (ensuring it remains above $\varphi \in C^{1,1}(\Omega)$) and decrease the energy. The fact that v is lower semi-continuous then follows from Proposition 3.3. We now prove that the set $\{v \leq \varphi\}$ is a closed set. First take a sequence $(x_k)_{k\in\mathbb{N}} \subset \{v \leq \varphi\}$ such that $x_k \to x \in \Omega$. Then by lower semi-continuity for v we have that $v(x) \leq \liminf_{k\to\infty} v(x_k) \leq \varphi$ and so $x \in \{v \leq \varphi\}$. It then follows that $\{v > \varphi\}$ is open. □

Now lemma 3.5 allows us to conclude that in addition to (2.3), v is harmonic in $\{v > \varphi\}$.

Theorem 3.6. *Let v be the minimiser of E over $\mathcal{A}_{\varphi,g}$, then v weakly solves*

$$\Delta v = 0 \text{ in } \{v > \varphi\}.$$

Proof. Since $\{v > \varphi\}$ is open, for any $x_0 \in \{v > \varphi\}$ we can find a ball $B_r(x_0) \subset\subset \{v > \varphi\}$. Moreover since $v - \varphi$ is lower semi-continuous it attains it's minimum on compact sets and hence the constant $c = \min_{\overline{B_r(x_0)}}(v - \varphi)$ is well defined and obviously positive. Then choose $\eta \in C_c^\infty(B_r(x_0))$ and note that for $|\varepsilon| \leq \frac{c}{\|\eta\|_{L^\infty}}$ we have that $v + \varepsilon\eta \in \mathcal{A}_{\varphi,g}$. By minimality it follows that

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \int_\Omega \nabla(v + \varepsilon\eta) \cdot \nabla(v + \varepsilon\eta) = 0$$

so that after differentiating we obtain

$$\int_\Omega \Delta v \eta = 0 \ \forall \eta \in C_c^\infty(B_r(x_0)).$$

□

3.3 Optimal Regularity

Now that we have Theorem 3.6 we know that v satisfies

$$\begin{cases} \Delta v \leq 0 & \text{in } \Omega, \\ \Delta v = 0 & \text{in } \{v > \varphi\}, \\ v \geq \varphi & \text{in } \Omega, \\ v = g & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

This allows us to prove the optimal regularity of v using the mean value formula for sub and super-harmonic functions. As we previously did we simplify the situation by subtracting the obstacle and considering $u = v - \varphi$. Note that (3.5) now becomes with $f = -\Delta\varphi \in L^\infty(\Omega)$

$$\begin{cases} \Delta u \leq f & \text{in } \Omega \\ \Delta u = f & \text{in } \{u > 0\} \\ u \geq 0 & \text{in } \Omega \\ u = g - \varphi & \text{on } \partial\Omega \end{cases}. \tag{3.6}$$

We first show that u separates at most quadratically from the free boundary and then use this to conclude that the second derivatives of u are bounded. We begin with the statement of quadratic growth that is suitable for this setting.

Lemma 3.7. *Let u be a solution to (3.6) then for any $x_0 \in \{u > 0\}$ with $dist(x_0, \partial\{u > 0\}) < \frac{1}{3} dist(x_0, \partial\Omega)$ there holds*

$$0 \leq u(x) \leq C\, dist(x, \partial\{u > 0\})^2,$$

with $C = C(n, \|f\|_{L^\infty})$.

Proof. We first pick some point $x \in \{u > 0\}$ with $\mathrm{dist}(x_0, \partial\{u > 0\}) < \frac{1}{3}\mathrm{dist}(x_0, \partial\Omega)$ and define $r = \mathrm{dist}(x_0, \partial\{u > 0\})$. Note that $B_r(x) \subset \{u > 0\}$ and this ball touches the free boundary, call this contact point y.

Define for $z \in \Omega$ the function $w(z) = u(z) + \|f\|_{L^\infty} \frac{|x-z|^2}{2n}$ and note that in $\{u > 0\}$ we have

$$\Delta w = \Delta u + \|f\|_{L^\infty} = f + \|f\|_{L^\infty} \geq 0,$$

and so w is sub-harmonic in this region. Using the mean value property at $x \in \{u > 0\}$ in the ball $B_r(x) \subset \{u > 0\}$ we obtain that

$$w(x) \leq \fint_{B_r(x)} u(z) + \|f\|_{L^\infty} \frac{|x - z|^2}{2n} \leq \fint_{B_r(x)} u(z) + \|f\|_{L^\infty} \frac{r^2}{2n}.$$

However $w(x) = u(x)$ and so we get

$$u(x) \leq \fint_{B_r(x)} u(z) + \|f\|_{L^\infty} \frac{r^2}{2n}. \tag{3.7}$$

Now define for $z \in \Omega$ the function $w(z) = u(z) - \|f\|_{L^\infty} \frac{|x-z|^2}{2n}$ and note that in Ω we have

$$\Delta w = \Delta u - \|f\|_{L^\infty} \le f - \|f\|_{L^\infty} \le 0,$$

and so w is super-harmonic. Using the mean value property at $y \in \partial\{u > 0\}$ in the ball $B_{2r}(y) \subset \Omega$ we obtain that

$$w(y) \ge \fint_{B_{2r}(y)} u(z) - \|f\|_{L^\infty} \frac{|x-z|^2}{2n} \ge \fint_{B_{2r}(y)} u(z) - \|f\|_{L^\infty} \frac{4r^2}{2n}.$$

However $w(y) = 0$ and so we get

$$\fint_{B_{2r}(y)} u(z) \le 4\|f\|_{L^\infty} \frac{r^2}{2n}. \tag{3.8}$$

Combining the inequalities (3.7) and (3.8) , and the facts that $B_r(x) \subset B_{2r}(y)$ and $u \ge 0$, we arrive at

$$\begin{aligned} u(x) - \|f\|_{L^\infty} \frac{r^2}{2n} &\le \fint_{B_r(x)} u(z) \\ &\le \frac{|B_{2r}(y)|}{|B_r(x)|} \fint_{B_{2r}(y)} u(z) \\ &\le 2^{n+2} \|f\|_{L^\infty} \frac{r^2}{2n}, \end{aligned}$$

which proves the claim. □

With the quadratic growth we can now prove the optimal regularity of u.

Theorem 3.8. *Let $u = v - \varphi$ where v satisfies (3.5), then $u \in C^{1,1}_{loc}(\Omega)$.*

Proof. We know that away from the free boundary u is smooth and so we just need to worry about what happens around $\partial\{u > 0\}$. Choose any $x \in \{u > 0\}$ with $\text{dist}(x, \partial\{u > 0\}) \le \frac{1}{3}\text{dist}(x, \partial\Omega)$ and define $r = \text{dist}(x, \partial\{u > 0\})$. The point here is that our right hand side is $f \in L^\infty$ and so we can't use Schauder estimates to get bounds on the second derivatives of u. However, we can use the following trick relying on the fact that $u = v - \varphi$. Define the function

$$w(y) = u(y) - \varphi(x) - \nabla\varphi(x) \cdot (y - x) \ \forall y \in B_r(x),$$

and note that $\Delta w = 0$ in $B_r(x)$. Then since $\varphi \in C^{1,1}(\Omega)$ we have for all $y \in B_r(x)$

$$\begin{aligned} |w(y) - u(y)| &= |\varphi(y) - \varphi(x) - \nabla\varphi(x) \cdot (y - x)| \\ &\le \frac{1}{2} \int_0^1 \left|D^2\varphi(tx + (1-t)y)\right| dt |x - y|^2 \\ &\le \frac{1}{2} \left\|D^2\varphi\right\|_{L^\infty(\Omega)} r^2 \end{aligned}$$

which when coupled with the quadratic growth of u yields

$$\|w\|_{L^\infty(B_r(x))} \le \left(\frac{1}{2}\left\|D^2\varphi\right\|_{L^\infty(\Omega)} + C\right) r^2.$$

Then applying interior estimates for harmonic functions to w in $B_r(x)$ we obtain

$$\frac{|Du(x)|}{r} + \left|D^2 u\right| \le C\left(n, \left\|D^2\varphi\right\|_{L^\infty(\Omega)}\right). \tag{3.9}$$

We first show that $u \in C^1(\Omega)$. We just need to show that u and Du can be extended continuously to zero across the free boundary. However this is true. Indeed by quadratic growth we have that $u(x) \le C\mathrm{dist}(x, \partial\{u > 0\})^2$ and by (3.9) we have that $|Du(x)| \le C\mathrm{dist}(x, \partial\{u > 0\})$ and so as $\mathrm{dist}(x, \partial\{u > 0\}) \to 0$ it follows that $u(x) \to 0$ and $Du(x) \to 0$. We can conclude that u is C^1 across the free boundary.

We now show that Du is locally Lipschitz across the free boundary. Define $\tilde{\Omega} = \left\{x \in \Omega : \mathrm{dist}(x, \partial\{u = 0\}) \le \frac{1}{6}\mathrm{dist}(x, \partial\Omega)\right\}$ and choose $x, y \in \tilde{\Omega}$. We have several cases to consider. The first is when both $x, y \in \{u = 0\}$, then the result is trivial. So from here we can assume $x \in \{u > 0\}$ and without loss of generality that

$$\mathrm{dist}(y, \partial\{u > 0\}) \le \mathrm{dist}(x, \partial\{u > 0\}).$$

The first case in this setting is when $x \in \{u > 0\}$ and $y \in \{u = 0\}$, which means that $|x - y| \ge \mathrm{dist}(x, \partial\{u > 0\})$. Using (3.9) and the fact that $Du(y) = 0$ we obtain that

$$\frac{|Du(x) - Du(y)|}{|x - y|} \le C\frac{\mathrm{dist}(x, \partial\{u > 0\})}{|x - y|} \le C.$$

The second case is when both $x, y \in \{u > 0\}$ which means that $|x - y| \le \mathrm{dist}(x, \partial\{u > 0\})$. Using once again (3.9) we obtain

$$|Du(x) - Du(y)| \le \int_0^1 \left|D^2 u\right|(tx + (1 - t)y)dt|x - y| \le C|x - y|.$$

Hence we have shown that u is $C^{1,1}$ across the free boundary and this concludes the proof. □

4 Free boundary regularity

In this section we begin our study of the free boundary for the obstacle problem

$$\begin{cases} \Delta u = f\chi_{\{u>0\}} & \text{in } B_1 \\ u \ge 0 & \text{in } B_1. \end{cases} \tag{4.1}$$

where $f \in C^{0,\alpha}(B_1)$. We begin by giving some first examples of solutions to the obstacle problem and then analysing their free boundaries.

4.1 First examples and non-degeneracy

In order to understand what sort of free boundaries could occur we will start by looking at various examples. In this first example we give an obvious construction of a solution in $\mathbb{R}^n$, $n \geq 3$ with right hand side $f(x) \equiv 1$, namely,

$$u(x) = (\frac{1}{2n}|x|^2 - |x|^{2-n} - C)_+,$$

where $C = -\left(\frac{1}{2n}((n-2)n)^{2/n} + ((n-2)n)^{(2-n)/n}\right)$. Here it is clear that $\Delta u = \chi_{\{u>0\}}$ and moreover, the free boundary is given by $\partial B_{r_n}(0)$ where $r_n = ((n-2)n)^{1/n}$. This is an example of a smooth free boundary since it is a sphere.

Some other obvious examples with smooth free boundaries in $\mathbb{R}^2$ are the solutions $u(x) = \frac{1}{2}(x_2)_+^2$ and $u(x) = \frac{1}{2}x_2^2$. Both of these solve the obstacle problem with $f = 1$. However we note a subtle difference between these two solutions, that is the shape of the contact set $\{u = 0\}$. In example $u(x) = \frac{1}{2}(x_2)_+^2$ the contact set is an entire half-space where as in the case $u(x) = \frac{1}{2}x_2^2$ it is only the line $x_2 = 0$.

It is actually quite hard to explicitly construct singularities. However, this was achieved by Schaeffer in [17] in which he constructed examples of free boundaries with cusps such as the one presented in figure 1 which was constructed in [12].

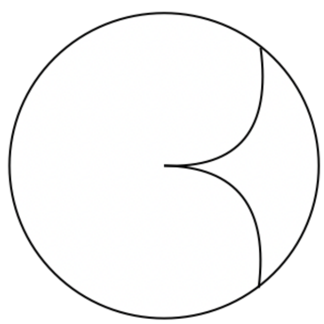

Figure 1: An example of a singularity in a free boundary

Moreover Schaeffer showed in [17] that given any two subsets of $\mathbb{R}^{n-1}$, $E \subset F \subset B_1$ with E open and F closed one can find a smooth super harmonic obstacle so that the zero level set of the solution will satisfy $E = \text{Int}(\{u = 0\}) \cap \mathbb{R}^{n-1}$ and $F = \{u = 0\} \cap \mathbb{R}^{n-1}$.

In order to actually study any regularity of the free boundary, we will need to make the necessary assumption $f \geq c_0 > 0$. As a consequence of this we have that close to the free boundary the solution grows at least quadratically. This is the content of the following proposition.

Proposition 4.1 (Non-degeneracy). *Let u be a solution to* (4.1) *and assume that*

$f \geq c_0 > 0$. Then for every free boundary point $x_0 \in \partial\{u > 0\} \cap B_{1/2}$ we have that

$$\sup_{B_r(x_0)} u \geq C(n, c_0) r^2, \tag{4.2}$$

for any $r \in (0, \frac{1}{2})$.

Proof. Fix some $x_0 \in \partial\{u > 0\}$ and an $r \in (0, \frac{1}{2})$. Now choose some $x_1 \in \{u > 0\}$ close to x_0 and define the function

$$w(x) = u(x) - \frac{c_0}{2n}|x - x_1|^2$$

and note that this is subharmonic in $\{u > 0\} \cap B_r(x_1)$ while $w(x_1) > 0$. The maximum principle then states that w must have a positive maximum on $\partial(\{u > 0\} \cap B_r(x_1))$. However, on the segment of this boundary that coincides with $\partial\{u > 0\}$ we have that $w < 0$ and so this positive maximum must occur on the segment $\partial B_r(x_1) \cap \{u > 0\}$. Hence, we obtain that

$$\sup_{\partial B_r(x_1)} w > 0,$$

and unravelling the definition of w we obtain

$$\sup_{\partial B_r(x_1)} u > \frac{c_0}{2n} r^2.$$

Sending $x_1 \to x_0$ and noting that $\sup_{B_r(x_0)} u \geq \sup_{\partial B_r(x_0)} u$ we obtain the result. □

This non-degeneracy property coupled with the quadratic growth proved in the previous sections actually yields that at all free boundary points x_0

$$0 < \frac{1}{C} r^2 \leq \sup_{B_r(x_0)} u \leq C r^2,$$

where C is a constant depending on n, c_0 and $\|f\|_{L^\infty}$. These properties are essential in our study of the free boundary as we will see in the following sections.

4.2 Overview of Results

In the previous section we saw two different types of free boundary points, points around which the free boundary is smooth and cusp points. Alternatively you can view these points as where the contact set is “thick” or “thin” around the free boundary. This notion is made precise in the following definition.

Definition 4.2 (Regular and Singular points)**.** *If $x_0 \in \partial\{u > 0\}$ satisfies*

$$\liminf_{r \to 0} \frac{|\{u = 0\} \cap B_r(x_0)|}{|B_r(x_0)|} > 0 \tag{4.3}$$

it is called a regular point.

If $x_0 \in \partial\{u > 0\}$ satisfies

$$\lim_{r \to 0} \frac{|\{u = 0\} \cap B_r(x_0)|}{|B_r(x_0)|} = 0 \tag{4.4}$$

it is called a singular point. The set of all regular points will be denoted as Reg(u) while the set of all singular points will be denoted by $\Sigma(u)$ (which we will often simply write as Σ).

It is important to put this definition into the context of the examples we saw in the previous subsection. First consider the case where $u(x) = \frac{1}{2}(x_2)_+^2$. Here the free boundary is made up of regular points. On the other hand according to Definition 4.2, the entire contact set $\{u = 0\}$ for $u(x) = \frac{1}{2}x_2^2$, which is its free boundary, is a line of singular points. However, it really isn't the same type of singularity as that in figure 1.

So it is not clear that this definition is helpful, or that it even categorises all possible types of free boundary points. The breakthrough of Caffarelli in [3] was in fact this very dichotomy, that the free boundary can be broken up into regular points and singular points in the sense of Definition 4.2, and that, around regular points the free boundary is smooth.

However, as our examples show, at least when $f \equiv 1$, we expect more; that there are some singular points where the free boundary is also smooth around it. In this direction, Caffarelli in [4] showed that Σ can be stratified appropriately so that each stratum can be covered by a C^1 manifold with an abstract dimensional modulus of continuity. Furthermore, in [5] they improved the qualitative C^1 regularity result to a quantitative $C^{1,\log^{\varepsilon_0}}$.

The results we expect from our examples were not proven until recently in [10] where it is showed, in the case $f \equiv 1$, that up to a set of "anomalous" points of higher codimension singular points can be covered by $C^{1,1}$ manifolds. In [11] it is showed that this regularity can be pushed to C^∞ in some cases.

It is important to note that all these results study the $\mathrm{Reg}(u)$ and $\Sigma(u)$ as two disjoint sets, that is, we do not really have a complete picture of the free boundary. The only case where we do have this picture is in $n = 2$ and $f \equiv 1$ where Sakai in [16] classified all the types of free boundaries one can get using analytic techniques.

The rest of this paper is dedicated to first understanding the, by now, classical Dichotomy theorem of Caffarelli (cf. Theorem 5.1) and present it's proof in the case of having a right hand side $f \in C^{0,\alpha}$. Then we move on to studying these recent covering results from [10] and adapt them to the case when $f \in C^{0,\alpha}$.

5 Classification of blow-ups

The main goal of this section is to introduce the blow-up technique from [3] to study the free boundary. From this we prove the dichotomy theorem of Cafarelli by classifying the possible blow-ups following the approach in [7] appropriately modified for the case when $f \in C^{0,\alpha}$.

5.1 Strategy

Let $u \in C^{1,1}(B_1)$ be a solution of (4.1) and $x_0 \in \partial\{u > 0\}$. We define the blow-up sequence of u at x_0 to be

$$u_{x_0,r}(x) = r^{-2}u(x_0 + rx),\ r > 0.$$

Note that the elements of the blow-up sequence $u_{x_0,r}$ satisfy the equation

$$\Delta u_{x_0,r}(x) = f(x_0 + rx).$$

Any possible limit of the blow-up sequence is then called a blow-up of u at x_0 and is denoted by u_{x_0}. If 0 is the free boundary point in question we will write $u_{0,r}(x) = u_r(x)$ for elements of the blow up sequence and u_0 for the blow-up of u at 0.

We can now state the dichotomy theorem of Caffarelli.

Theorem 5.1. *Let u be a solution to (4.1) and let x_0 be a free boundary point. Then*

- *if x_0 is a regular point then there exists $e \in \mathbb{S}^{n-1}$ such that*

$$u_{x_0}(x) = \lim_{r\downarrow 0} r^{-2}u(rx) = \frac{f(x_0)}{2}(x\cdot e)_+^2;$$

- *if x_0 is a singular point then there exists some $A \in \mathbb{R}^{n\times n}$ symmetric, positive definite with $\operatorname{tr}(A) = 1$ such that*

$$u_{x_0}(x) = \lim_{r\downarrow 0} r^{-2}u(rx)\frac{f(x_0)}{2}xAx^T$$

The first task in proving Theorem 5.1 is to identify any possible limits of these blow-up sequences and then classify them. We will not follow the original approach in [3] but rather we will classify blow-ups in the same way as done in [7], adapted to the case where $f \in C^{0,\alpha}$ for some $\alpha \in (0,1)$ using the Weiss energy introduced in [13].

We are now concerned with solutions $u \in C^{1,1}_{loc}(B_1)$ of

$$\begin{cases} \Delta u = f & \text{in } \{u > 0\} \\ u \geq 0 & \text{in } B_1 \\ 0 \in \partial\{u > 0\}, \end{cases} \tag{5.1}$$

where in order to simplify notation and to ensure we stay away from the fixed boundary we assumed that $0 \in \partial\{u > 0\}$.

Our first result is that the blow up sequences converge up to taking a subsequence.

Proposition 5.2. *Let u be a solution to (5.1) and let $r_k \downarrow 0$. Then, up to extracting a subsequence,*

$$u_{r_k} \to u_0 \text{ in } C^1_{loc}(\mathbb{R}^n)$$

where $u_0 \in C^{1,1}_{loc}(\mathbb{R}^n)$ satisfies

$$\begin{cases} \Delta u_0 = f(0) & \text{in } \{u_0 > 0\} \\ u_0 \geq 0 & \text{in } B_1. \end{cases} \tag{5.2}$$

Moreover, $0 \in \partial\{u_0 > 0\}$.

Proof. We first note that thanks to optimal regularity we have that

$$\left\| D^2 u_r \right\|_{L^\infty(B_{\frac{1}{r}})} \leq C, \tag{5.3}$$

and so $(u_{r_k})_{k\in\mathbb{N}}$ is a bounded sequence in $C^{1,1}_{loc}(\mathbb{R}^n)$. Therefore, the Arzela-Ascoli theorem gives that up to a subsequence (which we do not relabel),

$$u_{r_k} \to u_0 \text{ in } C^1_{loc}(\mathbb{R}^n)$$

for some $u_0 \in C^1_{loc}(\mathbb{R}^n)$. Letting $r_k \downarrow 0$ in (5.3) we automatically see that in fact $u_0 \in C^{1,1}_{loc}(\mathbb{R}^n)$. Similarly, since $u_r(x) \geq 0$ in B_1 we can pass to the limit and obtain that $u_0 \geq 0$ in B_1.

Fixing $K \subset \mathbb{R}^n$ compact, we choose $\eta \in H^1_0(K \cap \{u_0 > 0\})$ non-negative and note that after some $k \in \mathbb{N}$, $u_{r_k} > 0$ in $\{u_0 > 0\}$, that is, in the support of η. Since $\Delta u_{r_k} = f_{r_k}(x)$ in $\{u_{r_k} > 0\}$ we have that

$$\int_{\mathbb{R}^n} \nabla u_{r_k} \cdot \nabla \eta = -\int_{\mathbb{R}^n} f_{r_k} \eta,$$

and passing to the limit we see that

$$\int_{\mathbb{R}^n} \nabla u_0 \cdot \nabla \eta = -\int_{\mathbb{R}^n} f(0) \eta.$$

Since K was arbitrary, we conclude that this holds in $\{u_0 > 0\}$.

Finally to see that 0 is a free boundary point of u_0 we can pass to the limit in $u_{r_k}(0) = 0$ to see that indeed $u_0(0) = 0$. Now 0 is not contained in the zero level set $\{u_0 = 0\}$ since by non-degeneracy for any $\rho \in (0, 1/2)$ we have

$$\|u_r\|_{L^\infty(B_\rho(0))} = r^{-2} \|u\|_{L^\infty(B_{r\rho}(0))} \geq C r^{-2} (r\rho)^2 = \rho^2.$$

Again after passing to the limit we find $\|u_0\|_{L^\infty(B_\rho(0))} \geq \rho^2$ for all $\rho \in (0, 1/2)$ and so $0 \in \partial\{u_0 > 0\}$. □

We stress here that it is not clear whether blow-ups are unique. This subtle point will be returned to later after we classify blow-ups.

5.2 Classification of Blow-ups

The first interesting thing about blow-ups is that they are homogeneous of degree 2. We use the following fact about homogeneous functions.

Lemma 5.3. *A function $u \in C^1(\mathbb{R}^n)$ is k-homogeneous if and only if $x \cdot \nabla u - ku = 0$.*

Proof. Differentiating $u(\lambda x) = \lambda^k u(x)$ in λ and setting $\lambda = 1$ yields the forward implication. For the reverse direction note again by direct differentiation in λ and using $\lambda x \cdot \nabla u(x\lambda) - ku(x\lambda) = 0$ we obtain the ODE $\frac{d}{d\lambda}(u(\lambda x)) = \frac{k}{\lambda}u(\lambda x)$ which when solved with the initial condition $u(\lambda x) = u(x)$ at $\lambda = 1$ gives the result. □

Lemma 5.3 also gives one more handy fact about homogeneous functions: if u is k-homogeneous then the m^{th} derivative of u is $k - m$ homogeneous.

We now introduce the Weiss energy of a solution u at a point x_0 as

$$W(r, u, x_0; f(x)) = \frac{1}{r^{n+2}} \int_{B_r(x_0)} \left(|\nabla u|^2 + 2f(x)u\right) - \frac{2}{r^{n+3}} \int_{\partial B_r(x_0)} u^2. \tag{5.4}$$

When $x_0 = 0$ we will simply write $W(r, u; f)$ and when f is also clear by context we will omit it.

We have the following theorem from [13] establishing the almost monotonicity of $W(r, u, x_0; f(x))$.

Theorem 5.4. *There exists a continuous function $F(r)$ for $0 \le r \le 1$ with $F(0) = 0$ and a constant C depending on $\|u\|_{C^{1,1}}$, n and α such that $W(r, u, x_0; f(x)) + CF(r)$ is monotone non-decreasing for $r \in (0, 1/2)$.*

Proof. See Theorem M [13, Section2]. □

For the blow-up u_0 satisfying (5.2), we know that it's Weiss energy will be given by

$$W(r, u_0, x_0; f(0)) = \frac{1}{r^{n+2}} \int_{B_r(x_0)} \left(|\nabla u_0|^2 + 2f(0)u_0\right) - \frac{2}{r^{n+3}} \int_{\partial B_r(x_0)} u_0^2. \tag{5.5}$$

In this important case we actually have that W itself is monotone.

Proposition 5.5. *Let u be a solution of (5.2). Then the quantity (5.5) is non-decreasing for $r \in (0, 1)$.*

Proof. We assume we are centred at 0 and proceed by calculating the derivative of $W(r) := W(r, u; f(0))$. Note that we have the scaling

$$W(r) = \int_{B_1} |\nabla u_r|^2 + 2f(0)u_r - 2\int_{\partial B_1} u_r^2,$$

as well as

$$\frac{d}{dr}u_r = \frac{1}{r}\left(x \cdot \nabla u_r - 2u_r\right). \tag{5.6}$$

Differentiating we find

$$W'(r) = \int_{B_1} 2\nabla u_r \cdot \frac{d}{dr}(\nabla u_r) + 2f(0)\frac{d}{dr}u_r - 4\int_{\partial B_1} u_r \frac{d}{dr}u_r.$$

Exchanging $\frac{d}{dr}$ and ∇ and integrating by parts in the first term we see that

$$\begin{aligned}\int_{B_1} \nabla u_r \cdot \frac{d}{dr}(\nabla u_r) &= \int_{B_1} \nabla u_r \cdot \nabla(\frac{d}{dr}u_r) \\ &= -\int_{B_1} \Delta u_r \frac{d}{dr}u_r + \int_{\partial B_1} \nabla u \cdot \nu \frac{d}{dr}u_r.\end{aligned}$$

Observe that by (5.6) $\frac{d}{dr}u_r = 0$ in $\{u_r = 0\}$. Moreover, $\Delta u_r = f(0)$ in $\{u_r > 0\}$ and so we find that

$$\int_{B_1} \nabla u_r \cdot \frac{d}{dr}(\nabla u_r) = -\int_{B_1} f(0)\frac{d}{dr}u_r + \int_{\partial B_1} \nabla u \cdot \nu \frac{d}{dr}u_r.$$

Grouping terms and using the fact that the outward unit normal on the unit ball is simply x we find that

$$W'(r) = \frac{1}{r}\int_{\partial B_1} \left(x \cdot \nabla u_r - 2u_r\right)^2, \tag{5.7}$$

which is clearly non-negative. □

This result allows us to conclude the homogeneity of blow-ups.

Theorem 5.6 (Homogeneity of blow ups)**.** *Let u_0 be a solution of (5.2). Then u_0 is homogeneous of degree 2.*

Proof. In light of Lemma 5.3 and (5.7), we just need to show that $W(r, u_0; f(0))$ is constant. Indeed, if this is so then (5.7) yields

$$x \cdot \nabla u_0 - 2u_0 = 0.$$

To this end we note that by the scaling of W as well as Theorem 5.4

$$\begin{aligned}W(\rho, u_0; f(0)) &= \lim_{r\to 0} W(\rho, u_r; f(rx)) \\ &= \lim_{r\to 0} W(r\rho, u; f(x)) \\ &= W(0^+, u; f(x)).\end{aligned}$$

□

The cornerstone of the blow-up analysis is that blow-ups are convex.

Theorem 5.7. *Any blow-up u_0 satisfying (5.2) is convex.*

Proof. The strategy will be to show that $(\partial_{ee}u_0)_- = 0$ for any $e \in \mathbb{S}^{n-1}$.

To this end we first transfer information from u_0 to $\partial_{ee}u_0$ by considering for any small $t > 0$ the sequence of second order difference quotients

$$\delta^2_{te}u_0 = \frac{u_0(x+te) + u_0(x-te) - 2u_0(x)}{t^2}. \tag{5.8}$$

It is clear that in $\{u_0 > 0\}$ that $\Delta\delta^2_{te}u_0 \le 0$ while in $\{u_0 = 0\}$ we have that $\delta^2_{te}u_0 \ge 0$, hence $\min\{\delta^2_{te}u_0, 0\}$ is superharmonic. Moreover, by optimal regularity, $\delta^2_{te}u_0 \in C^{1,1}$ and is bounded uniformly in t.

Now note that $\min\{\delta^2_{te}u_0, 0\}$ converges pointwise to $\min\{\partial_{ee}u_0, 0\}$. Hence, by Lemma 3.4 we have that $\min\{\partial_{ee}u_0, 0\}$ is superharmonic, in particular, it is lower semi-continuous. Moreover, since u_0 is 2-homogeneous, $\min\{\partial_{ee}u_0, 0\}$ is 0-homogeneous and so the minimum must be attained at some $y_0 \in B_1$ (recall that a lower semi-continuous function achieves it's minimum on compact sets and that a 0-homogeneous function is radially constant).

However,

$$\fint_{B_r(y_0)} \min\{\partial_{ee}u_0, 0\}dx$$

is non-increasing in r and so $\min\{\partial_{ee}u_0, 0\}$ must be constant. Since $\min\{\partial_{ee}u_0, 0\} = 0$ on $\{u_0 = 0\}$ by the pointwise convergence, it follows that $\partial_{ee}u_0 \ge 0$. □

The first consequence of the convexity and homogeneity of blow-ups is the following classification of singular and regular points.

Proposition 5.8. *If 0 is a regular point, then $\{u_0 = 0\}$ has non-empty interior. If 0 is a singular point, then $\{u_0 = 0\}$ has empty interior.*

Proof. First suppose 0 is a regular point with a sequence $r_k \downarrow 0$ along which

$$\lim_{r_k \to 0} \frac{|\{u = 0\} \cap B_{r_k}|}{|B_{r_k}|} \ge \theta \tag{5.9}$$

for some $\theta > 0$. There exists a subsequence (not relabelled) under which $u_{r_k} \to u_0$ uniformly in B_1 by Proposition 5.2. Now if $\{u_0 = 0\}$ had empty interior, it would necessarily be contained in a hyperplane since it is a convex set. With no loss of generality suppose that $\{u_0 = 0\} \subset \{x_1 = 0\}$, then by continuity of u_0 and the fact that u_0 is positive outside $\{x_1 = 0\}$, for any $\delta > 0$ there is some $\varepsilon > 0$ such that $u_0 \ge \varepsilon$

in $\{|x_1| > \delta\} \cap B_1$. Then by uniform convergence of $u_{r_k} \to u_0$, there exists a $k \in \mathbb{N}$ such that $u_{r_k} \geq \frac{\varepsilon}{2}$ in $\{|x_1| > \delta\} \cap B_1$. This means that $\{u_{r_k} = 0\} \cap B_1 \subset \{|x_1| \leq \delta\} \cap B_1$. Hence,

$$\frac{|\{u_{r_k} = 0\} \cap B_1|}{|B_1|} \leq \frac{|\{|x_1| \leq \delta\} \cap B_1|}{|B_1|} \leq C\delta.$$

But $\{u_{r_k} = 0\} \cap B_1 = \{u = 0\} \cap B_{r_k}$, and so we have reached that

$$\frac{|\{u = 0\} \cap B_{r_k}|}{|B_{r_k}|} \leq C\delta.$$

Choosing $\delta = \frac{\theta}{2C}$ we have reached a contradiction to (5.9).

Now suppose 0 is a singular point and that $r_k \downarrow 0$ is a sequence along which

$$\lim_{r_k \to 0} \frac{|\{u_{r_k} = 0\} \cap B_1|}{|B_1|} = 0, \tag{5.10}$$

so up to extraction of a subsequence, $u_{r_k} \to u_0$ in $C^1(B_1)$. Since (5.10) gives that $\lim_{r_k \to 0} |\{u_{r_k} = 0 \cap B_1\}| = 0$ it immediately follows that $\Delta u_0 = f(0)$ in B_1. Indeed, we have that u_{r_k} satisfies

$$\int_{B_1} \nabla u_{r_k} \cdot \nabla \eta = \int_{B_1 \cap \{u_{r_k} > 0\}} f(r_k x),$$

and so passing to the limit using dominated convergence we obtain that $\Delta u_0 = f(0)$ in B_1. Now u_0 is non-negative, homogeneous which implies $u_0(0) = 0$, and convex implies that for every $e \in \mathbb{S}^{n-1}$, $\partial_{ee} u_0 \geq 0$. However, $\Delta u_0 = f(0)$ means that there is always at least one direction such that $\partial_{ee} u_0 > 0$. These facts immediately imply that u_0 always grows in some direction so that $\{u_0 = 0\}$ has empty interior. $\square$

Proposition 5.8 allows us to classify blow-ups at singular and regular points based on whether or not the contact set has empty or non-empty interior. In order to do this, we will first need three preliminary lemmas.

Lemma 5.9. *Suppose $H = \{x_1 = 0\}$ is a hyperplane and that $\Delta u = 1$ in $\mathbb{R}^n \backslash H$. If $u \in C^1(\mathbb{R}^n)$ then $\Delta u = 1$ in $\mathbb{R}^n$.*

Proof. Fix $R > 0$ and let $w \in C^1(B_R)$ satisfy

$$\begin{cases} \Delta w = 1 & \text{in } B_R \\ w = u & \text{on } \partial B_R. \end{cases}$$

Then $v = u - w \in C^1(B_R)$ satisfies

$$\begin{cases} \Delta v = 0 & \text{in } B_R \\ v = 0 & \text{on } \partial B_R. \end{cases}$$

We claim that $v \equiv 0$. To see this consider the 'tent' function $2R - |x_1|$ in B_R which is strictly positive and harmonic in B_R. Moreover note that the wedge is contained in $H = \{x_1 = 0\}$ and on ∂B_R this function is greater than or equal to R , in particular, it is greater than or equal to v on ∂B_R. Set $\kappa^* = \inf\{\kappa \geq 0 : v(x) \leq \kappa^*(2R-|x_1|),\ \forall x \in B_R\}$. Now if $\kappa^* > 0$ then there must exist some $p \in B_R$ such that $v(p) = \kappa^*(2R - |p_1|)$ and since $v \in C^1$ this point where v touches the 'tent' tangentially cannot be in the wedge, i.e. $p \in B_R \backslash H$. Now both v and $2R - |x_1|$ are harmonic in $B_R \backslash H$ and since two harmonic functions cannot touch at an interior point we reached a contradiction. Hence $\kappa^* = 0$ and $v \leq 0$. The argument repeated with $-v$ proves that $v \geq 0$ so that $v \equiv 0$. This proves the claim and the hence the lemma. $\square$

Lemma 5.10. *Suppose that Σ is a closed convex cone with vertex at the origin and $w \in C(\mathbb{R}^n)$ is a 1-homogenous function satisfying*

$$\begin{cases} \Delta w = 0 & \text{in } \Sigma^c, \\ w > 0 & \text{in } \Sigma^c, \\ w = 0 & \text{in } \Sigma. \end{cases}$$

Then Σ is a half space.

Proof. The proof is based on an application of the following version of the Hopf Lemma ([7, Lemma 1.15]).

Claim: Suppose $\Omega \subset \mathbb{R}^n$ satisfies the interior ball condition and define $d_\Omega(x) := \operatorname{dist}(x, \Omega^c)$. Furthermore, suppose $u \in C(\overline{\Omega})$ is harmonic and positive in $\Omega \cap B_2$ and $u \geq 0$ on $\partial\Omega \cap B_2$. Then there exists some $c_0 > 0$ such that $u \geq c_0 d_\Omega$ in $\Omega \cap B_1$.

Proof of Claim. Fix some $r \leq \frac{1}{2}$ let $h \in C(B_r)$ satisfy $\Delta h = 0$ in $B_r \backslash B_{r/2}$ with $h = 0$ on ∂B_r and and $h \equiv 1$ on $\overline{B_{r/2}}$. Now let $c_1 \leq \frac{1}{r}$ and note that $c_1(r - |x|) \leq 1$ in $B_{r/2}$. Moreover the maximum principle applied to $h(x) - c_1(r - |x|)$ in the annulus $B_r \backslash B_{r/2}$ shows that $h(x) \geq c_1(r - |x|)$ in the annulus. Hence we have that $h(x) \geq c_1(r - |x|)$ in all of B_r.

By the positivity of u in the interior of Ω we have that there exists some $c_2 > 0$ such that $u \geq c_2 > 0$ in $\{d_\Omega \geq \frac{r}{2}\}$. Now for any $x_0 \in \Omega$ such that $B_r(x_0) \subset \Omega$ we have that $u(x) \geq c_2 h(x_0 + x)$ in $B_r(x_0)$ by the same maximum principle argument above. This is then enough to conclude the claim with $c_0 = c_1 c_2$ since $h(x_0 + x) \geq c_1(r - |x - x_0|) \geq c_1 d_\Omega(x)$ in $B_r(x_0)$. $\square$

We now prove the lemma. Since Σ is convex there exists some $e \in \mathbb{S}^{n-1}$ so that the half space $H = \{x \cdot e > 0\}$ is contained in the complement of Σ, in particular $d_H \leq d_{\Sigma^c}$. Since the complement of a convex set satisfies the interior ball condition, we can use the claim in Σ^c to see that there exists some $c_0 > 0$ such that $w \geq c_0 d_{\Sigma^c}$ in $\Sigma^c \cap B_1$.

This can be extended to all of Σ^c by the 1-homogeneity of both w and d_{Σ^c}. As a consequence we have that

$$w \geq c_0 d_{\Sigma^c} \geq c_0 d_H \text{ in } \Sigma^c.$$

We define now

$$c_* = \sup\{c > 0 : \; w \geq c d_H \text{ in } \Sigma^c\}.$$

Note that $d_H(x) = (x \cdot e)_+$ and so $w - c_* d_H \geq 0$ is harmonic and $w - c_* d_H = 0$ on $\partial H \cap \partial \Sigma$. Now, suppose that $w - c_* d_H \not\equiv 0$ so that the strong maximum principle (applied in H) yields $w - c_* d_H > 0$ in H. Then we can apply the claim in H to find that $w - c_* d_H \geq c_0 d_H$. This implies that $w - (c_* + c_0) d_H \geq 0$, contradicting the definition of c_*.

It then follows that $w - c_* d_H \equiv 0$ and we obtain that w is a multiple of d_H. Since $d_H = 0$ outside of H, it follows that $\Sigma = H^c$, a half space. □

The following lemma regarding convex functions is also important.

Lemma 5.11. *Suppose $u : \mathbb{R}^n \to \mathbb{R}$ is convex and $\{u = 0\}$ contains the line $\{te' : t \in \mathbb{R}\}$ for some $e' \in \mathbb{S}^{n-1}$. Then $u(x + te') = u(x)$ for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$.*

Proof. With no loss of generality suppose $e' = e_n$ and we write $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$. Fix $x' \in \mathbb{R}^{n-1}$ and for any $\varepsilon > 0$, $M \in \mathbb{R}$ and $x_n \in \mathbb{R}$ we have by convexity applied to $((1-\varepsilon)x', x_n + \varepsilon M) = (1-\varepsilon)(x', x_n) + \varepsilon(0, x_n + M)$ that

$$u((1-\varepsilon)x', x_n + \varepsilon M) \leq (1-\varepsilon)u(x', x_n) + \varepsilon u(0, x_n + M) = (1-\varepsilon)u(x', x_n),$$

where we used the fact that $u(0, x_n + M) = 0$.

For any $\lambda \in \mathbb{R}$ set $M = \frac{\lambda}{\varepsilon}$ and sending $\varepsilon \to 0$ we obtain for any $\lambda \in \mathbb{R}$ and any $x_n \in \mathbb{R}$

$$u(x', x_n + \lambda) \leq u(x', x_n).$$

Choosing $\lambda = -x_n$ gives $u(x', 0) \leq u(x', -x_n)$ for all $x_n \in \mathbb{R}$. Choosing $x_n = 0$ we have that $u(x', \lambda) \leq u(x', 0)$ for all $\lambda \in \mathbb{R}$. In particular we have that $u(x', x_n) = u(x', 0)$ for all $x_n \in \mathbb{R}$ and so we have proved the claim. □

We can now classify blow-ups.

Proposition 5.12. *Let u be a solution to (5.1) and let u_0 be a blow-up at 0. Then either*

- $u_0 = \frac{f(0)}{2}(x \cdot e)_+^2$ *for some* $e \in \mathbb{S}^{n-1}$,
- *or* $u_0 = \frac{f(0)}{2} x A x^T$ *for some* $A \in \mathbb{R}^{n \times n}$ *symmetric positive definite with* $\mathrm{tr}(A) = 1$.

Proof. Suppose $\{u_0 = 0\}$ has non-empty interior. By homogeneity and convexity we have that $\{u_0 = 0\}$ is a closed convex cone with vertex at the origin.

We claim that the set $\{u_0 = 0\}$ is a half-space. First note that for some $\tau \in \mathbb{S}^{n-1}$ such that $-\tau \in \{u_0 = 0\}$ the function $w = \partial_\tau u_0$ is not identically zero. Moreover it holds that $w \geq 0$ in $\mathbb{R}^n$. To see this observe that for any $x \in \mathbb{R}^n$ by convexity we have that $\partial_{tt} u_0(x + t\tau) \geq 0$ and so $\partial_t u_0(x + t\tau)$ is monotone non-decreasing in t. Since for $t << -1$ we have that $\partial_t u_0(x + t\tau) = 0$, it follows that $\partial_t u_0(x + t\tau) \geq 0$ in $\mathbb{R}^n$ and so $w = \partial_\tau u_0 \geq 0$ in $\mathbb{R}^n$. Now w satisfies the assumptions of Lemma 5.10 and so it follows that $\{u_0 = 0\}$ is a half-space.

By Lemma 5.11 we have that u_0 then is a function of 1 variable. Hence $u_0(x) = U(x \cdot e)$ for some $e \in \mathbb{S}^{n-1}$ and $U \in C^{1,1}(\mathbb{R})$ satisfying $U''(t) = 1$, $U(0) = U'(0) = 0$ and $U \geq 0$. It then immediately follows that $U(t) = \frac{1}{2} t_+^2$ and so $u_0(x) = \frac{1}{2}(x \cdot e)_+^2$.

Now suppose $\{u_0 = 0\}$ has empty interior. Then by convexity $\{u_0 = 0\} \subset H$ where H is a hyperplane and so Lemma 5.9 tells us that $\Delta u_0 = f(0)$ in $\mathbb{R}^n$. Consequently, all the second derivatives of u_0 are harmonic. Moreover by the $C^{1,1}$ regularity, all the second derivatives are bounded and so by the Liouville theorem we obtain that the second derivatives of u_0 are constant. This immediately gives that u_0 is a quadratic polynomial. Furthermore, $u_0(0) = 0 = \nabla u_0(0)$ yields that $u_0(x) = \frac{f(0)}{2} x A x^T$ where $A \in \mathbb{R}^{n \times n}$ positive definite (since $u_0 \geq 0$) and $\mathrm{tr}(A) = 1$ (as $\Delta u_0 = f(0)$). $\square$

Now we may be tempted to say that our above proof coupled with Proposition 5.8 yields the proof of Theorem 5.1. However we still do not know that blow-ups are unique and so it is entirely possible that we converge to different blow-ups along different subsequences. However, what is clear from the above proof and Proposition 5.8 is that the *type of blow-up* is unique. That is, at regular points the blow-up will always be of the form $\frac{f(0)}{2}(x \cdot e)_+^2$ while at singular points we will always have a quadratic homogenous polynomial. The issue of uniqueness will be handled again later (and hence completing the proof of Theorem 5.1), however this classification is all we need to begin studying the free boundary. In fact, this discussion gives an alternative characterisation of regular and singular points.

Proposition 5.13 (Regular and Singular points)**.** *Let x_0 be a free boundary point. Then,*

- *x_0 is a singular point if and only if every blow-up at x_0 is a homogeneous quadratic polynomial,*
- *x_0 is a regular point if and only if every blow-up at x_0 is of the form $\frac{f(x_0)}{2}(x \cdot e)_+^2$ for some $e \in \mathbb{S}^{n-1}$.*

Proof. In each of these cases the forward implication is clear from the above discussion. As a consequence the converse implication is then also clear. Indeed if u_{x_0} is a homogeneous quadratic polynomial then we must necessarily have that x_0 is a regular point, or else u_{x_0} will be of the form $\frac{f(x_0)}{2}(x\cdot e)_+^2$, a contradiction. A similar argument establishes the second equivalence. □

5.3 Further consequences of the Weiss energy

Coupled with the Weiss energy, Proposition 5.13 gives yet another classification of singular and regular points. For the sake of notational convenience we introduce what is called the balanced energy of u at x_0

$$\omega(x_0) := \lim_{r\to 0} W(r, u, x_0; f(x)),$$

and we define the dimensional constant $\alpha_n = \frac{\mathcal{H}^{n-1}(\partial B_1)}{4n(n+2)}$.

Proposition 5.14. *Let x_0 be a free boundary point. Then;*

- *x_0 is a singular point if and only if $\omega(x_0) = f(x_0)^2\alpha_n$;*
- *x_0 is a regular point if and only if $\omega(x_0) = f(x_0)^2\frac{\alpha_n}{2}$.*

Proof. We first observe that $\omega(x_0) = W(1, u_{x_0}, 0; f(x_0))$ and after integrating by parts and using the 2-homogeneity of u_{x_0} we obtain that

$$W(1, u_{x_0}, 0; f(x_0)) = f(x_0)\int_{B_1} u_{x_0}(x)dx.$$

A direct computation shows that if $A \in \mathbb{R}^{n\times n}$ has $\mathrm{tr}(A) = 1$ then

$$f(x_0)\int_{B_1}\frac{f(x_0)}{2}x^T Axdx = f(x_0)^2\alpha_n,$$

and if $e \in \mathbb{S}^{n-1}$ then

$$f(x_0)\int_{B_1}\frac{f(x_0)}{2}(x\cdot e)_+^2 dx = f(x_0)^2\frac{\alpha_n}{2}.$$

These same computations show the converse implications as well. Indeed, if $\omega(x_0) = f(x_0)^2\alpha_n$ then there is no way for the blow-up u_{x_0} to be a half-space solution and so by Proposition 5.13 we obtain that u_{x_0} must be a quadratic polynomial and hence x_0 is a singular point. A similar argument establishes the second equivalence. □

For our purposes, the following result is the most important consequence of the balanced energy.

Proposition 5.15 (Balanced Energy)**.** *Given u a solution to (4.1), the balanced energy is an upper semi-continuous function of $x_0 \in \partial\{u > 0\}$. In particular, the set of singular points is a relatively closed subset of the free boundary.*

Proof. Let $\{x_k\}_{k\in\mathbb{N}}$ be a sequence of free boundary points that converge to $x_0 \in \partial\{u > 0\}$. We immediately have that

$$W(r, u, x_0; f(x)) = \lim_{k\to\infty} W(r, u, x_k; f(x)) \geq \lim_{k\to\infty} W(0^+, u, x_k, f(x)).$$

Then taking the limit as $r \to 0$ in the above yields that $\omega(\cdot)$ is upper semi-continuous.

We will show that the set of singular points is closed. Suppose $\{x_k\}_{k\in\mathbb{N}}$ is a sequence of singular points converging to some x_0. Then by upper semi-continuity and Proposition 5.14 we have that $\omega(x_0) \geq \lim_{k\to\infty} \omega(x_k) = \lim_{k\to\infty} f(x_k)^2\alpha_n$ and so we have that $\omega(x_0) = f(x_0)^2\alpha_n$. □

We conclude this section by introducing yet another useful characterisation of singular points, closely resembling Definition 4.2.

We first define the thickness function

$$\delta(r, u, x_0) = \frac{1}{r}\mathrm{mindiam}(\{u = 0\} \cap B_r(x_0)),$$

which measures the thickness of the contact set around a free boundary point x_0. Here mindiam is the infimum of the distances between two parallel planes enclosing A. It should come as no surprise that, in light of Definition 4.2, at a singular point x_0 we have that $\delta(r, u, x_0) \downarrow 0$ as $r \downarrow 0$. For our purposes we will need to show that for a reasonable class of solutions, the thickness function can be controlled by a universal modulus of continuity, that is, independent of u.

We introduce the class of solutions $P(M)$ which is the set of solutions to 4.1 with $\|u\|_{C^{1,1}} \leq M$ and $0 \in \partial\{u > 0\}$. We first have the following result from [13].

Proposition 5.16. *Let $u \in P(M)$ and $x_0 \in \partial\{u > 0\}$. Then for all $\varepsilon > 0$ there exist an $\eta_\varepsilon = \eta(\varepsilon, M, n, \|f\|_{C^{0,\alpha}})$ and $r_\varepsilon = r(\varepsilon, M, n, \|f\|_{C^{0,\alpha}})$ such that for any $0 < r \leq r_\varepsilon$*

$$W(r, u, x_0, f(x)) < f(x_0)^2\alpha_n - \varepsilon \implies \delta(r, u, x_0) > \eta_\varepsilon,$$

and

$$\delta(r, u, x_0) > \varepsilon \implies W(r, u, x_0, f(x)) \leq f(x_0)^2\alpha_n - \eta_\varepsilon.$$

Proof. See Proposition 1 from [13]. □

This proposition actually gives the following very useful result without having to prove any regularity of the free boundary around regular points.

Lemma 5.17. *Let $u \in P(M)$ and x_0 a singular point. There is a modulus of continuity $\sigma(r)$ depending only on M, $\|f\|_{C^{0,\alpha}}$ and n such that*

$$\delta(r, u, x_0) \leq \sigma(r).$$

Proof. Assume $x_0 = 0$ and we will write $\delta(r) := \delta(r, u, 0)$. Arguing by contradiction, we will suppose that no such σ exists. Let $\varepsilon > 0$ and suppose there exists a sequence $r_j \downarrow 0$ such that $\delta(r_j) > \varepsilon$. By Proposition 5.16 we have that for j large enough so that $r_j < r_\varepsilon$

$$W(r, u, x_0, f(x)) \leq f(x_0)^2 \alpha_n - \eta_\varepsilon,$$

in particular, $\omega(x_0) \leq f(x_0)^2 \alpha_n - \eta_\varepsilon$, and by Proposition 5.14 x_0 is a regular point, a contradiction. □

6 Regularity of the free boundary near regular points

For this section we will assume that $f \equiv 1$, i.e. we are considering solutions to

$$\begin{cases} \Delta u = \chi_{\{u>0\}} & \text{in } B_1 \\ u \geq 0 & \text{in } B_1 \, . \\ 0 \in \mathrm{Reg}(u). \end{cases} \tag{6.1}$$

We do this since the known results for any $f \in C^{0,\alpha}$ treat this case as a perturbation of the case when $f = 1$, see [1]. In fact, the results in [1] hold under the much weaker assumption that f is Dini continuous.

The approach we follow here is from [7, 14]. Recently in [15], the methods that we will present here were used to establish the regularity of the free boundary of solutions u to (4.1) under the additional assumption that $f \in W^{1,q}$ for some $q > n$. In particular, it is not treated as a perturbation of the case $f \equiv 1$.

6.1 Lipschitz Regularity

The approach we follow here is to first show that the free boundary is Lipschitz around regular points. The following simple observation is what will tie our classification of blow-ups at regular points with the main goal of this section.

Lemma 6.1. *Define the cone $C_\delta = \{x \in \mathbb{R}^n : x_n > \delta|x'|\}$ and write $x' = (x_1, \ldots, x_{n-1})$ and $B_1' = \{x' \in \mathbb{R}^{n-1} : |x'| \leq 1\}$. Let $u \in C^1(B_1)$ be a non-negative function such that*

$\partial_e u \ge 0$ for any $e \in C_\delta$. Then it follows that there exists a function $g : B_1' \to \mathbb{R}$ Lipschitz such that

$$\{u > 0\} = \{x \in B_1 : x_n > g(x')\},$$

with $\|g\|_{Lip(B_1)} \le \delta$. Namely, the boundary $\partial\{u > 0\}$ is a Lipschitz graph.

Proof. We define $g : B_1' \to \mathbb{R}$ as

$$g(x') = \{x_n : u(x', x_n) = 0 \text{ and } u(x', x_n + \varepsilon) > 0 \text{ for any } \varepsilon > 0\}. \tag{6.2}$$

Observe first that g is well defined. Indeed, if for some $x' \in B_1'$ there exists a, b with $a \ne b$ such that $a = g(x') = b$ then with no loss of generality we can assume $a > b$ and note that for $\varepsilon = a - b$ we have that $u(x', a) = u(x', b + (a - b)) > 0$. Hence, there cannot be more than one image of x' under g. In particular, $\Gamma_g := \{(x', x_n) : x_n = g(x')\}$ is a graph.

Now note that Γ_g can be touched above by the cone C_δ at each point $x' \in B_1'$. This will follow if we show that $u > 0$ in $(x', g(x')) + C_\delta$ for all $x' \in B_1$. If there existed a point $y \in (x', g(x')) + C_\delta$ such that $u(y) = 0$ then taking ε small enough $y \in (x', g(x') + \varepsilon) + C_\delta$ (note C_δ is an open cone). However, since $u(x', g(x') + \varepsilon) > 0$ and $\partial_e u \ge 0$ for any $e \in C_\delta$ we have that $u > 0$ in $(x', g(x') + \varepsilon) + C_\delta \ni y$. A contradiction.

This observation shows that g is indeed Lipschitz. Suppose that for $x, y \in \Gamma_g$ with $|x_n - y_n| > \delta|x' - y'|$. This would mean that $x - y \in C_\delta$ and so $x \in y + C_\delta$, so that x is not in Γ_g, a contradiction. Therefore, for any $x, y \in \Gamma_g$ we necessarily have that $|x_n - y_n| \le \delta|x' - y'|$. □

Remark 6.2. *Note that in the setting of Proposition 6.1 we have assumed for simplicity that $e = e_n$ and have defined the cone C_δ to be the cone generated by two rays with gradients $\pm\delta$. Taking $\gamma = \frac{\delta}{\sqrt{1+\delta^2}}$ we can see that $C_\delta = \{\tau \in \mathbb{S}^{n-1} : \tau \cdot e_n \ge \gamma\}$. We will switch to this way of representing cones and in particular, in what follows we can apply the result of Proposition 6.1 with a Lipschitz constant greater than $\left(\frac{1}{\gamma^2} - 1\right)^{-1/2}$.*

We can now see the usefulness of the blow ups. At regular points the blow ups satisfy for any $\tau \in \mathbb{S}^{n-1}$, $\partial_\tau u_0 = (x \cdot e)_+(x \cdot \tau)$ which is non-negative in the cone defined by $x \cdot e > \gamma$ for some $0 < \gamma < 1$. Note that this cone has an opening of $2\arccos(\gamma)$ and so as $\gamma \downarrow 0$ we will obtain that u_0 is monotone in the directions of a half plane.

Although we already know that the blow ups are half space solutions, and hence Lemma 6.1 is not so useful, the key idea is that we are able to, with a little bit of work, transfer this qualitative information to the rescaling u_r (for an appropriate r to be chosen) and then just by undoing the scaling, we will be able to show Lipschitz regularity of the free boundary around regular points.

Now to transfer this information to some rescaling, say u_{r_0}, we will require the following lemma.

Lemma 6.3. *For any $r_0 > 0$ let $\Omega = \{u_{r_0} > 0\}$, and $N_\delta = \{x \in B_1 : dist(x, \partial\Omega) < \delta\}$. For some $\tau \in \mathbb{S}^{n-1}$ we define $w = \partial_\tau u_{r_0}$ and note that w satisfies*

$$\begin{cases} \Delta w = 0 & \text{in } \Omega \cap B_1 \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.3}$$

Furthermore, assume that $w \geq -c_1$ in N_δ and $w \geq C_2 > 0$ in $\Omega \backslash N_\delta$ for some constants $c_1, C_2 > 0$.

Then, if c_1/C_2, $\delta > 0$ are small enough, $w > 0$ in $B_{1/2} \cap \Omega$.

Proof. Since $w > 0$ in $(\Omega\backslash N_\delta) \cap B_{1/2}$ by assumption, we only need to prove that $w \geq 0$ in $N_\delta \cap \Omega \cap B_{1/2}$. Suppose that this was not the case, namely, that there existed some $y_0 \in N_\delta \cap \Omega \cap B_{1/2}$ with $w(y_0) < 0$. For some $\kappa > 0$ to be determined later, we define on $B_{1/4}(y_0)$ the function

$$v(x) = w(x) - \kappa\left(u_{r_0}(x) - \frac{1}{2n}|x - y_0|^2\right).$$

Since v is harmonic in $B_{1/4}(y_0) \cap \Omega$ and $v(y_0) < 0$ we have that v must attain a negative minimum on the boundary of $B_{1/4}(y_0) \cap \Omega$. However, for appropriate values of c_1, C_2 and $\kappa > 0$ we can show that this is impossible.

First consider the section of the boundary that coincides with $\partial\Omega$. We have by assumption that $w = 0$ there and that $u_{r_0} = 0$ there also. Hence, $v > 0$ with no conditions on our constants.

Now consider the component of the boundary within N_δ, i.e. $\partial B_{1/4} \cap N_\delta$. By quadratic growth and our assumptions we have that

$$v \geq -c_1 - C\kappa\delta^2 + \frac{\kappa}{2n}\frac{1}{16}.$$

On the final component of the boundary, which lies outside N_δ we have by optimal regularity of u_{r_0} that

$$v \geq C_2 - C\kappa.$$

It is now clear that if $\delta \leq \left(\frac{1}{32nC}\right)^{1/2}$ and the constants κ, c_1 and C_2 satisfy

$$\frac{c_1}{\left(\frac{1}{32n} - C\delta^2\right)} \leq \kappa \leq \frac{C_2}{C},$$

we can arrange for $\inf_{\partial(B_{1/4}(y_0) \cap \Omega)} v \geq M$ for some $M > 0$. □

Using Lemma 6.3, we can transfer information of the blow-up on to u_{r_0} which is the content of the following proposition.

Proposition 6.4. *For any $\gamma > 0$ there exists an $r_0 = r_0(\gamma)$ such that for any $0 < r \le r_0$ and $\tau \in \mathbb{S}^{n-1}$ with $\tau \cdot e \ge \gamma$*

$$\partial_\tau u_{r_0} \ge 0 \text{ in } B_{1/2}.$$

Proof. As a consequence of the C^1_{loc} convergence and our classification of blow ups, for any $\varepsilon > 0$ (to be determined later) there exists some $e = e(\varepsilon) \in \mathbb{S}^{n-1}$ and $r_\varepsilon := r_\varepsilon(\varepsilon) > 0$ such that for any $0 < r_0 \le r_\varepsilon$ there holds

$$\left| u_{r_0}(x) - \frac{1}{2}(x \cdot e)_+^2 \right| \le \varepsilon \text{ in } B_1 \tag{6.4}$$

and

$$|\partial_\tau u_{r_0}(x) - (x \cdot e)_+ (x \cdot \tau)| \le \varepsilon \text{ in } B_1. \tag{6.5}$$

We first begin by showing that the free boundary $\partial\Omega = \partial\{u_{r_0} > 0\} \subset \{|x \cdot e| \le C_0\sqrt{\varepsilon}\}$ for some C_0 only depending n, where here we continue with the notation $\Omega = \{u_{r_0} > 0\}$. Indeed, if $x \cdot e > C_0\sqrt{\varepsilon}$ then by (6.4) we have that

$$u_{r_0} > \frac{1}{2}(C_0\sqrt{\varepsilon})^2 - \varepsilon > 0, \tag{6.6}$$

if $C_0^2 > 2$. On the other hand, if there existed a free boundary point $x_0 \in \{x \cdot e < -C_0\sqrt{\varepsilon}\}$, then by non-degeneracy we will have that

$$\sup_{B_{C_0\sqrt{\varepsilon}}(x_0)} u_{r_0} > c(n)(C_0\sqrt{\varepsilon})^2 > 2\varepsilon, \tag{6.7}$$

if $c(n)C_0^2 > 2$. This contradicts (6.4) since $(x \cdot e)_+ = 0$ in this region. Clearly (6.6) and (6.7) hold for $C_0 > \max\{2^{1/2}, (2/c(n))^{1/2}\}$ and thus C_0 only depends on n.

Now for any $\tau \in \mathbb{S}^{n-1}$ with $\tau \cdot e > \gamma$ we let $w = \partial_\tau u_{r_0}$ and note that

- w is bounded and harmonic in $\Omega \cap B_1$;
- $w = 0$ on $\partial\Omega \cap B_1$.

In order to apply Lemma 6.3 we need to find some appropriate lower bounds for w in N_δ and then in $(\Omega \backslash N_\delta) \cap B_1$. We have by (6.5) that in B_1, and in particular N_δ,

$$\partial_\tau u_{r_0} \ge -\varepsilon + \gamma(x \cdot e)_+ \ge -\varepsilon.$$

Now consider $x \in (\Omega \backslash N_\delta) \cap B_1$ and we would like to get a crude estimate for $x \cdot e$. Note that the distance between x and the free boundary is greater than δ. On the other hand, the free boundary is contained in the strip $|x \cdot e| \le C_0\sqrt{\varepsilon}$ and hence is at most distance $C_0\sqrt{\varepsilon}$ above or below the line $\{x \cdot e = 0\}$. From this we can conclude that $x \cdot e > \delta - C_0\sqrt{\varepsilon}$ and so by (6.5)

$$\partial_\tau u_{r_0} \ge -\varepsilon + \gamma(x \cdot e)_+ > -\varepsilon + \gamma(\delta - C_0\sqrt{\varepsilon}) > 0,$$

if $\varepsilon = \varepsilon(\gamma)$ is chosen sufficiently small enough. The result then follows immediately from Lemma 6.3 in which we take r_0 to be any $0 < r_0 \le r_\varepsilon$ for $\varepsilon = \varepsilon(\gamma)$ as above. □

Now that we have transferred the cone condition to u_{r_0}, we just need to note that after setting $\gamma = \frac{1}{2}$ and scaling with $r_0 = r_0(\frac{1}{2})$, we have that $\partial_\tau u \geq 0$ in $B_{\frac{r_0}{2}}$ for all $\tau \in \mathbb{S}^{n-1}$ with $\tau \cdot e \geq \frac{1}{2}$. Lemma 6.1 then gives that the free boundary is a Lipschitz graph in $B_{\frac{r_0}{2}}$.

As an additional consequence of Proposition 6.4 we have the uniqueness of blow-ups at regular points.

Corollary 6.5. *Let u be a solution of (6.1) and 0 be a regular free boundary point. Then u_0 is unique.*

Proof. Suppose that for a sequence $(r_k)_{k\in\mathbb{N}}$, the blow up sequence converges to $u_0(x) = \frac{1}{2}(x \cdot e)_+^2$ along a subsequence r_{k_l}. Now fix some $\gamma > 0$ and let $r_0(\gamma)$ be that given by Proposition 6.4. Now suppose that for a different subsequence r_{k_j} the blow up is $u_0'(x) = \frac{1}{2}(x \cdot e')_+^2$. Then we have that for any $r_{k_j} < r_0(\gamma)$ and $\tau \in \mathbb{S}^{n-1}$ such that $\tau \cdot e \geq \gamma$

$$\partial_\tau u'_{r_{k_j}}(x) \geq 0 \quad \text{in } B_{1/2}.$$

This immediately implies that $\tau \cdot e' \geq 0$ for all τ such that $\tau \cdot e \geq \gamma$. Sending $\gamma \to 0$ we see that $e = e'$ and hence the blow-up is unique. □

This in fact completes the proof of Caffarelli's dichotomy theorem in the case of regular points.

6.2 $C^{1,\alpha}$ regularity via Boundary Harnack Principle

From here we would like to prove that the free boundary is $C^{1,\alpha}$ for some $\alpha > 0$ which is the content of the following theorem.

Theorem 6.6. *Let u be a solution of (6.1). Then there exists some $r_0 > 0$ and $\alpha > 0$ small such that $\partial\{u > 0\}$ is $C^{1,\alpha}$ in $B_{\frac{r_0}{4}}$.*

The key tool will be the following Boundary Harnack principle.

Theorem 6.7. *Let Ω be a Lipschitz domain and suppose that w_1 and w_2 are two positive harmonic functions in $B_1 \cap \Omega$ vanishing on $\partial\Omega \cap B_1$. Moreover, suppose there exists some $C_\circ$ such that $C_\circ^{-1} \leq \|w_i\|_{L^\infty(B_{1/2})} \leq C_\circ$ for $i = 1, 2$. Then, there exists some $C = C(n, C_\circ, \Omega)$ and $\alpha = \alpha(n, C_\circ, \Omega) > 0$ small such that*

$$\left\| \frac{w_1}{w_2} \right\|_{C^{0,\alpha}(\overline{\Omega} \cap B_{1/4})} \leq C.$$

Proof. See Theorem 5.39 in [7]. □

We can now give the

Proof of Theorem 6.6. Since the blow-up is unique we can assume without any loss of generality that $e = e_n$ so that the blow-up at 0 is $u_0(x) = \frac{1}{2}x_n^2$. Moreover for any $\gamma > 0$ there exists some $r_0 = r_0(\gamma) > 0$ (given by Proposition 6.4) and $g : \mathbb{R}^{n-1} \to \mathbb{R}$ Lipschitz so that $\partial\{u > 0\} = \{x_n = g(x')\}$ in $B_{\frac{r_0}{2}}$.

Now define $\Omega = \{u_{r_0} > 0\}$ (which has Lipschitz boundary g in $B_{1/2}$) and the functions $w_n = \partial_n u_{r_0}$ and for $i = 1, \dots, n-1$ define $w_i = \partial_i u_{r_0} + \partial_n u_{r_0}$. Note that $\partial_i + \partial_n = \sqrt{2}\partial_\tau$ where $\tau \cdot e_n = \frac{1}{\sqrt{2}}$ and so by Proposition 6.4 we have that $w_i \geq 0$ in $B_{1/2}$ for $i = 1, \dots, n$, and in particular, for all $\varepsilon > 0$ we have that $w_i + \varepsilon > 0$.

Applying Theorem 6.7 to $w_n + \varepsilon$ and $w_i + \varepsilon$ for any $i = 1, \dots, n-1$ and letting $\varepsilon \downarrow 0$ we obtain that

$$\left\| \frac{w_i}{w_n} \right\|_{C^{0,\alpha}(\overline{\Omega} \cap B_{1/4})} \leq C$$

for some $\alpha > 0$. Since $\frac{w_i}{w_n} = 1 + \frac{\partial_i u_{r_0}}{\partial_n u_{r_0}}$ we obtain that

$$\left\| \frac{\partial_i u_{r_0}}{\partial_n u_{r_0}} \right\|_{C^{0,\alpha}(\overline{\Omega} \cap B_{1/4})} \leq C,$$

that is the quotient $\frac{\partial_i u_{r_0}}{\partial_n u_{r_0}}$ is $C^{0,\alpha}$ up to the free boundary $\partial\{u_{r_0} > 0\}$ in $B_{1/4}$. Now, given any $t > 0$ the normal vector to the level set $\{u_{r_0}(x) = t\}$ has components

$$\nu_i = \frac{\frac{\partial_i u_{r_0}}{\partial_n u_{r_0}}}{\sqrt{1 + \sum_{j=1}^{n-1} \left(\frac{\partial_j u_{r_0}}{\partial_n u_{r_0}} \right)^2}},$$

which is $C^{0,\alpha}$. Now, taking $t \downarrow 0$ we obtain that the normal vector to the free boundary is $C^{0,\alpha}$ in $B_{r_0/4}$, which proves the result. ☐

7 The structure of the Singular Set

In this section we will study the structure of the singular set of solutions to (4.1). We have followed here the methods in [9–11, 14], sometimes with minor modifications.

7.1 Monotonicty Formulae

At this point it will be convenient to change our notation. Namely, from now on, $w_r := w(r\cdot)$. Moreover we fill fix the following sets,

$$M = \{\text{symmetric } n \times n \text{ matrices with } tr(A) = 1\},$$

$$P = \{p(x) = x^T A x : \quad A \in M\}.$$

Before introducing the monotonicity formulas we define the following dimensionless quantities,

$$D(r,w) := r^{2-n} \int_{B_r} |\nabla w|^2 = D(1, w_r),$$

and

$$H(r,w) := r^{1-n} \int_{\partial B_r} w^2 = H(1, w_r).$$

It will be useful to compute the derivatives of these quantities at $r = 1$ and then rescale back to arbitrary r. To see how the derivatives scale we use the fact that

$$\frac{D(r+h,w) - D(r,w)}{h} = \frac{D(1+\frac{h}{r}, w_r) - D(1,w_r)}{r \cdot \frac{h}{r}},$$

to conclude that $D'(r,w) = \frac{1}{r} D'(1,w_r)$.

We can now introduce the following version of the Weiss energy

$$W(r,w) = r^{-4} \{D(r,w) - 2H(r,w)\},$$

which in this setting is also almost monotone. For the next two lemmas we will suppose we are centred at $0 \in \Sigma$. These lemmas are taken from [11, Appendix A].

Lemma 7.1 (Weiss Almost Monotonicity)**.** *There exists some $C > 0$ depending only on n and $\|f\|_{C^{0,\alpha}}$ and such that for all $p \in P$ and any $r \in (0,1)$*

$$\frac{d}{dr} W(r, u - f(0)p) \geq -Cr^{\alpha-1}. \tag{7.1}$$

Proof. We compute first the derivative of W at $r = 1$ for any $w \in C^{1,1}(B_1)$ and obtain

$$W'(1,w) = D'(1,w) - 2H'(1,w) - 4\{D(1,w) - 2H(1,w)\}.$$

Now using integration by parts we compute that

$$\begin{aligned}
D'(1,w) &= \frac{d}{dr}\Big|_{r=1} r^2 \int_{B_1} |\nabla w|^2 (r\cdot) \\
&= 2D(1,w) + \sum_{i,j} \int_{B_1} 2 w_i x_j w_{ij} \\
&= 2D(1,w) + 2\sum_{i,j} \left(\int_{\partial B_1} w_i x_j w_j \nu_i - \int_{B_1} (w_i x_j)_i w_j \right) \\
&= 2D(1,w) + 2\int_{\partial B_1} w_\nu^2 - 2\int_{B_1} \Delta w (x \cdot \nabla w) - 2\int_{B_1} w_i \delta^i_j w_j \\
&= 2D(1,w) + 2\int_{\partial B_1} w_\nu^2 - 2\int_{B_1} \Delta w (x \cdot \nabla w) - 2D(1,w) \\
&= 2\int_{\partial B_1} w_\nu^2 - 2\int_{B_1} \Delta w (x \cdot \nabla w).
\end{aligned}$$

We also have that
$$H'(1,w) = 2\int_{\partial B_1} w w_\nu$$
and
$$D(1,w) = \int_{B_1} |\nabla w|^2 = \int_{\partial B_1} w(x\cdot\nabla w) - \int_{B_1} w\Delta w.$$
We therefore obtain that
$$\begin{aligned}W'(1,w) &= 2\int_{\partial B_1} w_\nu^2 - 2\int_{B_1}\Delta w(x\cdot\nabla w) - 4\int_{\partial B_1} w w_\nu - 4\left\{\int_{\partial B_1} w(x\cdot\nabla w) - \int_{B_1} w\Delta w - 2\int_{\partial B_1} w^2\right\}\\ &= 2\int_{B_1}(2w - x\cdot\nabla w)\Delta w + 2\int_{\partial B_1}(w_\nu - 2w)^2.\end{aligned}$$
Now to scale back we observe
$$\begin{aligned}W'(r,w) &= -4r^{-5}\left\{D(r,w) - 2H(r,w)\right\} + r^{-4}\left\{D'(r,w) - 2H'(r,w)\right\}\\ &= -4r^{-5}\left\{D(1,w_r) - 2H(1,w_r)\right\} + r^{-5}\left\{D'(1,w_r) - 2H'(1,w_r)\right\}\\ &= r^{-5}W'(1,w_r).\end{aligned}$$
Consequently we obtain
$$W'(r,w) = \frac{1}{r^5}W'(1,w_r),$$
and so we have a lower bound
$$W'(r,w) \geq \frac{2}{r^5}\int_{B_1}(2w_r - x\cdot\nabla w_r)\Delta w_r. \tag{7.2}$$
Now for the specific case $w = u - f(0)p$ we observe that
$$\Delta w_r = r^2(f_r\chi_{\{u_r>0\}} - f(0))$$
and by exploiting the fact that $f \in C^{0,\alpha}$ we have
$$\left|\Delta w_r + r^2 f_r\chi_{\{u_r=0\}}\right| \leq Cr^{2+\alpha}.$$
We can now complete our estimate as
$$\begin{aligned}\int_{B_1}(2w_r - x\cdot\nabla w_r)\Delta w_r &= \int_{B_1}(2w_r - x\cdot\nabla w_r)(\Delta w_r + r^2 f_r\chi_{\{u_r=0\}}) - \int_{B_1}(2w_r - x\cdot\nabla w_r)(r^2 f_r\chi_{\{u_r=0\}})\\ &\geq -Cr^{2+\alpha}\int_{B_1}|2w_r - x\cdot\nabla w_r| + 2r^2\int_{B_1\cap\{u_r=0\}}(2p_r - x\cdot\nabla p_r)f_r\\ &\geq -Cr^{4+\alpha}\end{aligned}$$
where in the last line we used the quadratic growth of w as well as the fact that p is 2-homogenous. Substituting this back into (7.2) we obtain the result. □

The following is the adaptation of the Monneau monotonicity formula.

Lemma 7.2 (Monneau Almost Monotonicity)**.** *There exists some $C > 0$ depending only on n and $\|f\|_{C^{0,\alpha}}$ such that for all $p \in P$ and any $r \in (0,1)$*

$$\frac{d}{dr} r^{-4} H(r, u - f(0)p) \geq -Cr^{\alpha-1}.$$

As a consequence, blow ups are unique at singular points.

Proof. We first observe that for any $w \in C^{1,1}(B_1)$ we have that

$$\begin{aligned}
\frac{d}{dr}(r^{-4}H(r,w)) &= -4r^{-5}H(r,w) + r^{-4}H'(r,w) \\
&= r^{-5}\left(-4H(1,w_r) + H'(1,w_r)\right) \\
&= \frac{2}{r}\left(W(r,w) + r^{-4}\int_{B_1} w_r \Delta w_r\right).
\end{aligned}$$

Let $p \in P$ and choose a subsequence for which $r_k^{-2}u_{r_k} \to f(0)q$ where $f(0)q$ is a blow-up at 0. Using Lemma 7.1 we have that

$$\begin{aligned}
W(0^+, u - f(0)p) &= \lim_{r\to 0} W(r, u - f(0)p) \\
&= \lim_{r_k \to 0} W(r_k, u - f(0)p) \\
&= \lim_{r_k \to 0}\left(D(1, r_k^{-2}u_{r_k} - f(0)p) - 2H(1, r_k^{-2}u_{r_k} - f(0)p)\right) \\
&= (D(1, f(0)q - f(0)p) - 2H(1, f(0)q - f(0)p)) \\
&= f(0)^2\left[\int_{B_1} |\nabla(q-p)|^2 - 2\int_{\partial B_1}(q-p)^2\right] \\
&= f(0)^2\left[-\int_{B_1}\Delta(q-p)(q-p) + \int_{\partial B_1}(x\cdot\nabla(q-p) - 2(q-p))^2\right] \\
&= 0.
\end{aligned}$$

Now we can integrate (7.1) and obtain that $W(r, u - f(0)p) \geq -Cr^{\alpha}$. This allows us to complete the estimate

$$\begin{aligned}
\frac{d}{dr}(r^{-4}H(r,w)) &= \frac{2}{r}\left(W(r,w) + r^{-4}\int_{B_1} w_r\Delta w_r\right) \\
&\geq \frac{2}{r}\left(-Cr^{\alpha} + r^{-4}\int_{B_1} w_r \Delta w_r\right) \\
&= \frac{2}{r}\left(-Cr^{\alpha} + r^{-4}\int_{B_1} w_r r^2(f_r - f(0)) + r^{-4}\int_{B_1\cap\{u_r=0\}} p_r r^2 f(0)\right) \\
&\geq -Cr^{\alpha-1}.
\end{aligned}$$

Now to see that this implies that blow-ups at singular points are unique we first prove the following claim.

Claim: Let $F : (0,1) \to \mathbb{R}$ be non-negative and monotone non-decreasing. If $F(r_j) \to 0$ for a sequence $r_j \to 0$, then it follows that $F(r_l) \to 0$ for any sequence $r_l \to 0$.

Proof. Suppose there existed another sequence $(\rho_k)_{k\in\mathbb{N}}$ such that $F(\rho_k) \to c > 0$. Fix $j \in \mathbb{N}$ large enough so that $F(r_j) < \frac{c}{2}$. Now pick k large enough so that $\rho_k < r_j$. By monotonicity we have

$$\frac{c}{2} > F(r_j) \geq F(\rho_k) > c,$$

a contradiction. □

Note that if $f(0)q$ is the blow-up along a subsequence r_k then we have that

$$\lim_{r_k \to 0} r_k^{-2} H(r_k, u - f(0)q) \to 0,$$

and so the claim yields that

$$\lim_{r \to 0} r^{-2} H(r, u - f(0)q) \to 0.$$

Consequently every convergent subsequence must converge to $f(0)q$ and the blow-up is unique. □

From now on the blow-up at a singular point x_0 will be represented as $f(x_0)p_{x_0}$ for a certain $p_{x_0} \in P$. Recall $P(M)$ is the set of solutions to (4.1) with $\|u\|_{C^{1,1}} \leq M$ and $0 \in \partial\{u > 0\}$. The following are essentially Lemmas 7.3 and 7.7 from [14] with minor adaptations for our case.

Lemma 7.3. *Let $u \in P(M)$ and let 0 be a singular point. There exists a modulus of continuity $\sigma(r)$ depending only on n, M and $\|f\|_{C^{0,\alpha}}$ such that for any $0 < r < 1$ there exists a $q^r \in P$ such that*

$$\|u - f(0)q^r\|_{L^\infty(B_{2r})} \leq \sigma(r) r^2,$$

and

$$\|\nabla u - f(0)\nabla q^r\|_{L^\infty(B_{2r})} \leq \sigma(r) r.$$

Proof. Let $\varepsilon > 0$ and suppose there exists sequences $r_j \downarrow 0$ and $\{u_j\}_{j\in\mathbb{N}} \subset P(M)$ with

$$\|u - f(0)q\|_{L^\infty(B_r)} > \varepsilon r^2 \ \ \forall q \in P,$$

or

$$\|\nabla u - f(0)\nabla q\|_{L^\infty(B_r)} > \varepsilon r, \ \ \forall q \in P.$$

Now considering the blow-up sequence $v_j = r_j^{-2}(u_j)_{r_j}(x)$ we have that $v_j \to v_0$ in $C^{1,\alpha}_{loc}(\mathbb{R}^n)$ up to subsequences. By Lemma 5.17 we have that for any $\rho > 0$

$$\delta(\rho, r_j^{-2}(u_j)_{r_j}, 0) = \delta(\rho r_j, u_j, 0) \leq \sigma(\rho r_j) \to 0 \text{ as } r_j \downarrow 0.$$

Hence $\Delta v_0 = f(0)$ a.e. in $\mathbb{R}^n$ and by the classification of blow-ups $v_0 = f(0)p$ for some $p \in P$. By assumption we then have that

$$\|v_j - v_0\|_{L^\infty(B_1)} > \varepsilon,$$

or

$$\|\nabla v_j - \nabla v_0\|_{L^\infty(B_1)} > \varepsilon,$$

which contradicts the convergence of $v_j \to v_0$ in $C^{1,\alpha}_{loc}$. □

Lemma 7.3 combined with Monneau monotonicity yields the following crucial result.

Proposition 7.4. *Let $u \in P(M)$, x_0 a singular point and denote by $f(x_0)p_{x_0}$ the blow up of u at x_0. There exists a modulus of continuity $\sigma(r)$ depending only on n, M and $\|f\|_{C^{0,\alpha}}$ such that for any $x_0 \in \Sigma \cap \overline{B_{1/2}}$ and $x \in B_1$*

$$|u(x) - f(x_0)p_{x_0}(x - x_0)| \le \sigma(|x - x_0|)|x - x_0|^2,$$

and

$$|\nabla u(x) - f(x_0)\nabla p_{x_0}(x - x_0)| \le \sigma(|x - x_0|)|x - x_0|.$$

Moreover for any $x_1, x_2 \in \Sigma \cap \overline{B_{1/2}}$ we have

$$\|f(x_1)p_{x_1} - f(x_2)p_{x_2}\|_{L^2(\partial B_1)} \le \sigma(|x_1 - x_2|).$$

Proof. Let σ be as in Lemma 7.3. Now let $\varepsilon > 0$ and let $r_\varepsilon > 0$ such that $\sigma(r) \le \varepsilon$ for all $0 < r \le r_\varepsilon$. With no loss of generality assume $r_\varepsilon^\alpha \le \frac{C_n}{C}\varepsilon^2$ where C is the constant from Lemma 7.2. Moreover, for $0 < r \le r_\varepsilon$ let q^r be as in Lemma 7.3 and with no loss of generality assume $x_0 = 0$. Now by Monneau monotonicity we have that

$$\begin{aligned}
\int_{\partial B_1} (r^{-2}u_r - f(0)q^{r_\varepsilon})^2 + Cr^\alpha &= r^{-n-3}\int_{\partial B_r} (u - f(0)q^{r_\varepsilon})^2 + Cr^\alpha \\
&\le r_\varepsilon^{-n-3}\int_{\partial B_{r_\varepsilon}} (u - f(0)q^{r_\varepsilon})^2 + Cr_\varepsilon^\alpha \\
&\le C_n\varepsilon^2 + Cr_\varepsilon^\alpha \\
&\le C_n\varepsilon^2.
\end{aligned}$$

Taking the limit as $r \to 0$ along an appropriate subsequence we obtain

$$\int_{\partial B_1} (f(0)p_0 - f(0)q^{r_\varepsilon})^2 \le C_n\varepsilon^2. \tag{7.3}$$

Note that this is the $L^2(\partial B_1)$ norm on the finite dimensional vector space of quadratic homogeneous polynomials and so is equivalent to the C^1 norm which gives

$$|f(0)p_0 - f(0)q^{r_\varepsilon}| + |f(0)\nabla p_0 - f(0)\nabla q^{r_\varepsilon}| \le C_n\varepsilon.$$

Combining this with Lemma 7.3 we immediately obtain that in B_{r_ε}

$$|u - f(0)p_0| \leq |u - f(0)q^{r_\varepsilon}| + |f(0)q^{r_\varepsilon} - f(0)p_0| \leq C\varepsilon r_\varepsilon^2$$

and

$$|\nabla u - f(0)\nabla p_0| \leq |\nabla u - f(0)\nabla q^{r_\varepsilon}| + |f(0)\nabla q^{r_\varepsilon} - f(0)\nabla p_0| \leq C\varepsilon r_\varepsilon.$$

For the last part of the proposition we can assume that $x_2 = 0$. Now as above we have that

$$\begin{aligned}
\int_{\partial B_1} (r^{-2}u_{x_1,r} - f(x_1)q^{r_\varepsilon})^2 + Cr^\alpha &= r^{-n-3}\int_{\partial B_r(x_1)} (u - f(x_1)q^{r_\varepsilon}(x-x_1))^2 + Cr^\alpha \\
&\leq r_\varepsilon^{-n-3}\int_{\partial B_{r_\varepsilon}(x_1)} (u - f(x_1)q^{r_\varepsilon}(x-x_1))^2 + Cr_\varepsilon^\alpha \\
&\leq C_n\varepsilon^2 + Cr_\varepsilon^\alpha \\
&\leq C_n\varepsilon^2.
\end{aligned}$$

Now by letting $r \to 0$ along a subsequence we have

$$\int_{\partial B_1} (f(x_1)p_{x_1} - f(x_1)q^{r_\varepsilon})^2 \leq C_n\varepsilon^2.$$

This coupled with (7.3) yields

$$\begin{aligned}
\int_{\partial B_1} (f(x_1)p_{x_1} - f(0)p_0)^2 &\leq 2\left(\int_{\partial B_1} (f(x_1)p_{x_1} - f(x_1)q^{r_\varepsilon})^2 + \int_{\partial B_1} (f(0)p_0 - f(0)q^{r_\varepsilon})^2\right) \\
&\leq C_n\varepsilon^2.
\end{aligned}$$

□

7.2 Stratification and C^1 Regularity of the singular set

We begin this section by stratifying the singular set, which is possible thanks to the uniqueness of the blow-ups since we can assign to each $x_0 \in \Sigma$ a unique $p_{x_0} \in P$.

We define the sets

$$\Sigma_m = \{x_0 \in \Sigma : \ \dim(\ker(p_{x_0})) = m\},$$

where $m = 0, 1, \cdots, n-1$ and we have that the singular set is decomposed as

$$\Sigma = \Sigma_0 \cup \cdots \cup \Sigma_{n-1}.$$

We can now state the structure theorem of Caffarelli.

Theorem 7.5. *Σ_m is locally contained in a C^1 manifold of dimension m.*

The proof of this result requires the Whitney extension theorem which we state here without proof.

Theorem 7.6 (Whitney's Extension Theorem)**.** *Let E be a compact subset of $\mathbb{R}^n$ and $h : E \to \mathbb{R}^n$ a map. Suppose that for any $x_0 \in E$ there exists a polynomial P_{x_0} of degree m such that*

- $P_{x_0}(x_0) = h(x_0)$
- $|D^k P_{x_0}(x_1) - D^k P_{x_1}(x_1)| = o(|x_1 - x_0|^{m-k})$ *for all $x_1, x_0 \in E$ and $k = 0, \cdots, m$.*

Then h extends to a C^m function on $\mathbb{R}^n$ such that

$$f(x) = P_{x_0}(x) + O(|x - x_0|^m),$$

for all $x_0 \in E$.

Proof of Theorem 7.5. Let K be any compact subset of B_1 and since Σ is closed, $E = \Sigma \cap K$ is compact. Then for any $x_0 \in E$ we let $P_{x_0} = f(x_0)p_{x_0}(x - x_0)$, the blow-up at x_0 and we let h be the constant zero function.

Clearly, $P_{x_0}(x_0) = h(x_0) = 0$ for all $x_0 \in E$. The next condition is the content of Proposition 7.4. Indeed for any free boundary point $u(x) = \nabla u(x) = 0$ and after noticing that $P_{x_1}(x_1) = \nabla P_{x_1}(x_1) = 0$ we can conclude the second condition for $k = 0$ and $k = 1$ immediately. The condition for $k = 2$ is also handled in Proposition 7.4 as it is equivalent to the continuity of the map $x \mapsto p_x$.

The Whitney extension theorem therefore gives $F \in C^2(\mathbb{R}^n)$ with $F \equiv 0$ on E. Moreover, we have that

$$E \subset \{\nabla F = 0\} = \bigcap_{i=1}^{n} \{\partial_{x_i} F = 0\}.$$

Now suppose $x_0 \in \Sigma_m$. Up to re-arranging the co-ordinate axes, we can have that the non-zero eigenvalues of $D^2 F(x_0) = D^2 p_{x_0}$ are $e_1, \ldots, e_{n-m}$ and so

$$\det\left(D^2_{e_1,\ldots,e_{n-m}} F(x_0)\right) \neq 0.$$

The implicit function theorem then gives that $\bigcap_{i=1}^{m} \{\partial_{x_i} F = 0\}$ is an m-dimensional C^1 manifold around x_0. Since $\Sigma_m \cap K \subset E$ the result follows. $\square$

7.3 Almgren Frequency Formula

In order to gain more information about the regularity of the singular set, around singular points we will perform a second order blow-up, namely we will blow up the function $u - f(x_0)p_{x_0}$. As in the last section we will assume $0 \in \Sigma$ and suppose $f(0)p_0$

is the blow-up of u at 0. Now we will consider $w = u - f(0)p_0$ and consider possible limits as $r \downarrow 0$ of the renormalised second order blow-up sequence

$$\tilde{w}_r(x) = \frac{w_r}{\|w_r\|_{L^2(\partial B_1)}}.$$

As usual, our first task is to classify the possible blow-ups. However, our classification here will involve understanding the limiting values of the frequency functional

$$\phi(r,w) = \frac{D(r,w)}{H(r,w)}.$$

We will denote these possible limiting values as $\lambda_* := \lim_{r\downarrow 0} \phi(r,w)$. One important feature of the frequency as $r \downarrow 0$ is that it constrains the growth in the power scale $H(r,w) \sim r^{2\lambda_*}$. Roughly speaking, we expect from Proposition 3.4 in [9] that

$$\frac{H(R,w)}{H(r,w)} \sim \left(\frac{R}{r}\right)^{2\lambda_*} \text{ for } 0 < r < R << 1.$$

This is not quite true, see Lemma 7.8 for the rigorous statement, but is the intuition behind why classifying the frequency is useful. This is formally saying that the frequency controls the growth of the $L^2(\partial B_1)$ norm of w. We already know from Monneau monotonicity that $H(r,w) \sim r^4$ as $r \downarrow 0$, however what the frequency allows us to conclude is that perhaps this r^4 power could be improved to something higher. To see exactly how this useful, we will present the following from [10] which they have proved for the case $f = 1$ (note we will state and prove an analogous result for our purposes in Proposition 7.17).

Proposition 7.7. *Let $n \geq 2$, $m \in \{1, 2, \ldots, n-1\}$ and $\lambda > 2$. Let $l \in \mathbb{N}$ and $\beta \in (0,1]$ satisfy $l + \beta = \lambda$ and define*

$$S_{m,\lambda} := \{x_0 \in \Sigma_m : \ \phi(0^+, u(x_0 + \cdot) - p_{x_0}) \geq \lambda\}.$$

Then $S_{m,\lambda}$ is locally contained in a m-dimensional manifold of $C^{l-1,\beta}$.

This tells us that for singular points with $\lambda_* > 2$, the manifold covering them is actually more regular than C^1 in the Hölder scale. We will see that it is not possible to classify the possible values of λ_* at *every* singular point using our methods, however, this will present no problem in terms of covering the strata with manifolds with regularity better than C^1.

The rest of this section will be dedicated to collecting the various monotonicity formulae that we will need in our analysis. In order to deal with the errors introduced by the Hölder right hand side we will use the truncated frequency

$$\phi^\gamma(r,w) = \frac{D(r,w) + \gamma r^{2\gamma}}{H(r,w) + r^{2\gamma}}$$

where $\gamma \in (2,3)$ will be the truncation parameter that we will fix later.

Note, it is not true that $\lim_{r\downarrow 0} \phi^\gamma(r,w) = \lim_{r\downarrow 0} \phi(r,w)$. In order to understand when this is true, we will need the following lemma from [9].

Lemma 7.8. *Let $R \in (0,1)$ and $w : B_R \to [0,\infty)$ be a $C^{1,1}$ function. Assume that for some $\kappa \in (0,1)$ we have the almost monotonicity of the truncated frequency*

$$\frac{d}{dr}\phi^\gamma(r,w) \geq \frac{2}{r}\frac{\left(r^{2-n}\int_{B_r} w\Delta w\right)^2}{\left(H(r,w)+r^{2\gamma}\right)^2} - r^{\kappa-1} \qquad \forall r \in (0,R).$$

Then the following holds:

a. Suppose $0 < \underline{\lambda} \leq \phi^\gamma(r,w) \leq \overline{\lambda}$ for all $r \in (0,R)$. Then for any $\delta > 0$ we have

$$\frac{1}{C_\delta}\left(\frac{R}{r}\right)^{2\underline{\lambda}-\delta} \leq \frac{H(R,w)+R^{2\gamma}}{H(r,w)+r^{2\gamma}} \leq C_\delta\left(\frac{R}{r}\right)^{2\overline{\lambda}+\delta}$$

for all $r \in (0,R)$ where C_δ depends only on $n, \gamma, \kappa, \overline{\lambda}, \delta$.

b. If in addition

$$\frac{r^{2-n}\int_{B_r} w\Delta w}{H(r,w)+r^{2\gamma}} \geq -r^\kappa \qquad \forall r \in (0,R)$$

then for $\lambda_* := \phi^\gamma(0^+,w)$ we have

$$\exp\left(-4/\kappa^2\right)\left(\frac{R}{r}\right)^{2\lambda_*} \leq \frac{H(R,w)+R^{2\gamma}}{H(r,w)+r^{2\gamma}}.$$

Proof. See Lemma 4.1 from [9]. □

We will now gather some very important remarks.

Remark 7.9. *a. A consequence of Lemma 7.8 is that $\phi^\gamma(0^+,w) \leq \gamma$ for w as in Lemma 7.8(a). Indeed, suppose that $\phi^\gamma(0^+,w) > \gamma$ so that for some $\delta > 0$ small we could choose $\underline{\lambda} = \gamma + \delta$ and so for small r, by the almost monotonicity, we will have that $\phi^\gamma(r,w) \geq \underline{\lambda} > \gamma$. However Lemma 7.8(a) will then yield that $r^{2\gamma} \leq Cr^{2\gamma+\delta}$ which cannot be true for small r.*

b. A further consequence is that if $\phi^\gamma(0^+,w) < \gamma$ then $\frac{r^{2\gamma}}{H(r,w)} \downarrow 0$ as $r \downarrow 0$. Suppose that $\phi^\gamma(0^+,w) < \gamma$, then for r small enough we have that $\phi^\gamma(r,w) \leq \gamma - \beta$ for some small β. Then applying Lemma 7.8(a) we would have that $H(r,w) \geq Cr^{2\gamma-2\beta}$ so that $\frac{r^{2\gamma}}{H(r,w)} \leq Cr^{2\beta}$.

c. *As a consequence of Remark 7.9(b) we have that if* $\phi^\gamma(0^+,w) < \gamma$ *then* $\lambda_* := \phi(0^+,w) = \phi^\gamma(0^+,w)$ *by just letting* $r \downarrow 0$ *in*

$$\phi^\gamma(r,w) = \frac{\phi(r,w) + \gamma\frac{r^{2\gamma}}{H(r,w)}}{1 + \frac{r^{2\gamma}}{H(r,w)}}.$$

d. *In the setting of Lemma 7.8 we know that the limit* $\phi^\gamma(0^+,w)$ *exists by almost monotonicity. We can distinguish two important cases when studying this limit:*

   i. *The first case is when* $H(r,w) \le C_\sigma r^{2\gamma-\sigma}$ *for all* $\sigma \in (0,1)$ *for some* $C_\sigma > 0$ *that depends on* σ. *In this case we cannot conclude any information about the limit* $\phi(0^+,w)$ *other than that it is greater than or equal to* γ. *This is because in this case* $\phi^\gamma(0^+,w) = \gamma$, *that is, the truncation is 'active'. Indeed, if* $\phi^\gamma(0^+,w) < \gamma$ *the exact same argument from Remark 7.9(b) also yields that* $H(r,w) \ge Cr^{2\gamma-\beta}$ *contradicting our assumption.*

   ii. *The other case is when the condition* $H(r,w) \le C_\sigma r^{2\gamma-\sigma}$ *fails for some* $\sigma \in (0,1)$ *and some sequence* $r_k \downarrow 0$. *In this case we have that* $\frac{r_k^{2\gamma}}{H(r_k,w)} \le C_\sigma r_k^{2\sigma} \downarrow 0$ *as* $r_k \downarrow 0$ *and so we have that* $\lambda_* := \phi(0^+,w) = \phi^\gamma(0^+,w)$. *Moreover, in the setting of Lemma 7.8(b) we can also conclude that* $\phi^\gamma(0^+,w) < \gamma$. *Suppose that* $\phi^\gamma(0^+,w) = \gamma$, *then Lemma 7.8(b) would imply that* $H(r_k,w) \le r_k^{2\gamma}$ *and this coupled with the assumption yields*

   $$C_\sigma r_k^{2\gamma-\sigma} \le H(r_k,w) \le r_k^{2\gamma},$$

   a contradiction. Furthermore, since $\phi^\gamma(0^+,w) < \gamma$, *by Remark 7.9(b) we have that* $\frac{r^{2\gamma}}{H(r,w)} \downarrow 0$ *as* $r \downarrow 0$ *(not just for the specific subsequence* r_k*).*

e. *Formally, this Remark is saying that as* $r \downarrow 0$ *we have that*

$$\phi^\gamma(r,w) \sim \min\{\phi(r,w),\gamma\}.$$

With this in mind, we must now investigate the possible monotonicity of the ϕ^γ for some appropriate γ. For this we will need another lemma from [9].

Lemma 7.10. *Let* $w \in C^{1,1}(B_1)$. *Then for* $r \in (0,1)$ *we have that*

$$\frac{d}{dr}\phi(r,w) \ge \frac{2}{r}\frac{(r^{2-n}\int_{B_r} w\Delta w)^2 + E(r,w)}{H(r,w)^2}$$

and

$$\frac{d}{dr}\phi^\gamma(r,w) \ge \frac{2}{r}\frac{(r^{2-n}\int_{B_r} w\Delta w)^2 + E^\gamma(r,w)}{(H(r,w)+r^{2\gamma})^2},$$

where

$$E(r,w) = \left(r^{2-n}\int_{B_r} w\Delta w\right) D(r,w) - \left(r^{2-n}\int_{B_r}(x\cdot\nabla w)\Delta w\right) H(r,w)$$

and

$$E^\gamma(r,w) = \left(r^{2-n}\int_{B_r} w\Delta w\right)\left(D(r,w)+\gamma r^{2\gamma}\right) - \left(r^{2-n}\int_{B_r}(x\cdot\nabla w)\Delta w\right)\left(H(r,w)+r^{2\gamma}\right).$$

Proof. See Lemma 2.3 from [9]. □

Using this we can prove the almost monotonicity of the truncated frequency for some $\gamma > 2$. These next lemmas are taken from [11].

Lemma 7.11. *For $\gamma = 2 + \frac{\alpha}{8}$ and $\varepsilon = \frac{\alpha}{2}$ there exists some $C > 0$ depending only on n, $\|f\|_{C^{0,\alpha}}$ and $\|u\|_{C^{1,1}}$ such that for $w = u - f(0)p$ for any $p \in P$, it holds for any $r \in (0,1)$*

$$\phi^\gamma(r,w) \geq 2 - Cr^\epsilon, \tag{7.4}$$

and

$$\frac{d}{dr}\phi^\gamma(r,w) \geq -Cr^{\epsilon-1}. \tag{7.5}$$

As a consequence we also have that

$$\phi^\gamma(r,w) \leq C. \tag{7.6}$$

Moreover, there holds

$$\frac{\int_{B_1} w_r\Delta w_r}{H(r,w)+r^{2\gamma}} \geq -Cr^\varepsilon. \tag{7.7}$$

Proof. Since $\gamma > 2$ we have that

$$\begin{aligned}\phi^\gamma(r,w) - 2 &= \frac{D(r,w)+\gamma r^{2\gamma}}{H(r,w)+r^{2\gamma}} - 2\\ &= \frac{D(r,w) - 2H(r,w) + (\gamma-2)r^{2\gamma}}{H(r,w)+r^{2\gamma}}\\ &\geq \frac{r^4 W(r,w)}{H(r,w)+r^{2\gamma}}\\ &\geq -Cr^{\alpha-2\gamma+4}\end{aligned}$$

where in the last line we used Lemma 7.1. This establishes (7.4) as $2\gamma - 4 = \frac{\alpha}{4}$ and so the estimate holds using any $\varepsilon \leq \frac{3\alpha}{4}$ (in particular $\varepsilon = \frac{\alpha}{2}$ works).

By Lemma 7.10 we have

$$\frac{d}{dr}\phi^\gamma(r,w) \geq \frac{2}{r}\int_{B_1}\frac{(\phi^\gamma(r,w)w_r - x\cdot\nabla w_r)\Delta w_r}{H(r,w)+r^{2\gamma}},$$

and so setting $\lambda_r := \phi^\gamma(r, w)$ we estimate the above integral in a similar way as in the proof of Lemma 7.1 to obtain

$$\begin{aligned}\int_{B_1}(\lambda_r w_r - x\cdot\nabla w_r)\Delta w_r &\geq r^2\int_{B_1\cap\{u_r=0\}}(\lambda_r f(0)p_r - x\cdot\nabla f(0)p_r)) - Cr^{2+\alpha}\int_{B_1}|\lambda_r w_r - x\cdot w_r|\\ &= r^4(\lambda_r - 2)f(0)\int_{B_1\cap\{u_r=0\}} pf_r - r^{4+\alpha}(\lambda_r+1)\\ &\geq -Cr^4(r^{\frac{3\alpha}{4}} - r^\alpha(\lambda_r+1)),\end{aligned}$$

where in the last line we used (7.4) with ε replaced by $\frac{3\alpha}{4}$. In the end we obtain that

$$\lambda_r' \geq -Cr^{3+\frac{3\alpha}{4}-2\gamma}(\lambda_r+1) \tag{7.8}$$

showing that $\log(\lambda_r+1)$ is almost monotone. Moreover, integrating (7.8) from r to 1 and using optimal regularity of u we have that λ_r is bounded from above and hence shows (7.6). This observation improves (7.8) and we obtain

$$\lambda_r' \geq -Cr^{3+\frac{3\alpha}{4}-2\gamma} = -Cr^{\varepsilon-1}$$

establishing (7.5). For (7.7) recall from the proof of Lemma 7.2 that

$$\int_{B_1} w_r\Delta w_r \geq -Cr^{4+\alpha}$$

and so we immediately obtain

$$\frac{\int_{B_1} w_r\Delta w_r}{H(r,w)+r^{2\gamma}} \geq \frac{-Cr^{4+\alpha}}{r^{2\gamma}} \geq -Cr^{\frac{7\alpha}{8}} \geq -Cr^\varepsilon.$$

□

We conclude this subsection by introducing the functional

$$H^\gamma_\lambda(r,w) = r^{-2\lambda}(H(r,w)+r^{2\gamma})$$

for $\lambda > 0$. The following consequence of Lemma 7.11 gives the almost monotonicity of the functional H^γ_λ.

Corollary 7.12. *For all $0 \leq \lambda \leq \phi^\gamma(0^+, w)$*

$$\frac{d}{dr}H^\gamma_\lambda(r,w) \geq -Cr^{\varepsilon-1}, \quad r\in(0,1),$$

Proof. By (7.7) we have that

$$\frac{d}{dr}\log\big(H^\gamma_\lambda(r,w)\big) \geq \frac{2}{r}(\phi^\gamma(r,w)-\lambda) + \frac{2}{r}\frac{\int_{B_1} w_r\Delta w_r}{H(r,w)+r^{2\gamma}} \geq -Cr^{\varepsilon-1},$$

as $\lambda \leq \phi^\gamma(0^+,w) \leq \phi^\gamma(r,w) + Cr^\varepsilon$ by monotonicity of the quantity $\phi^\gamma(r,w)+Cr^\varepsilon$. Integrating this we obtain that $H^\gamma_\lambda(r,w)$ is bounded from above. Since

$$H^\gamma_\lambda(r,w)\frac{d}{dr}\log\big(H^\gamma_\lambda(r,w)\big) = \frac{d}{dr}H^\gamma_\lambda(r,w),$$

we obtain the result. □

7.4 Second order blow-ups analysis

We will now classify the second order blow-ups, this is our analogue of Proposition 2.10 from [10]. Here we are essentially following again [9–11].

Proposition 7.13. *Let* 0 *be a singular point,* $w = u - f(0)p_0$ *and for* $r > 0$ *we define*

$$\tilde{w}_r = \frac{w_r}{\|w_r\|_{L^2(\partial B_1)}}.$$

Furthermore, define $L = \{p_0 = 0\}$, $m = \dim(L)$ *and fix* $\gamma = 2 + \frac{\alpha}{8}$. *Then:*

a. For $m = n-1$ *either*

  i. For all $\sigma \in (0,1)$ *there exists some* $C_\sigma > 0$ *such that for all* $r \in (0,1)$, $H(r,w) \le C_\sigma r^{2\gamma-\sigma}$ *and hence* $\phi^\gamma(0^+,w) = \gamma$ *or;*

  ii. For every sequence $r_k \downarrow 0$ *there exists a subsequence* r_{k_l} *such that* $\tilde{w}_{r_{k_l}} \rightharpoonup q$ *in* $W^{1,2}(B_1)$, *as well as locally uniformly in* B_1, *and* $\lambda_* := \phi(0^+,w) = \phi^\gamma(0^+,w) \ge 2+\alpha_0$ *for some dimensional constant* $\alpha_0 > 0$. *Moreover* q *is a* λ_**-homogeneous solution of the Signorini problem with* 0 *obstacle on* L *which is*

$$\begin{cases} \Delta q \le 0, \;\; q\Delta q = 0 & \text{in } \mathbb{R}^n, \\ \Delta q = 0 & \text{in } \mathbb{R}^n \backslash L, \\ q \ge 0 & \text{in } L. \end{cases} \tag{7.9}$$

b. For $m \le n-2$ *either*

  i. For all $\sigma \in (0,1)$ *there exists some* $C_\sigma > 0$ *such that for all* $r \in (0,1)$, $H(r,w) \le C_\sigma r^{2\gamma-\sigma}$ *and hence* $\phi^\gamma(0^+,w) = \gamma$ *or;*

  ii. For every sequence $r_k \downarrow 0$ *there exists a subsequence* r_{k_l} *such that* $\tilde{w}_{r_{k_l}} \rightharpoonup q$ *in* $W^{1,2}(B_1)$ *where* q *is a* 2*-homogeneous harmonic polynomial, in particular* $\lambda_* := \phi(0^+,w) = \phi^\gamma(0^+,w) = 2$.

This result will require several lemmas to prove.

In order to exclude that λ_* cannot be equal to 2 in case $a(ii)$ we will need to show some structural conditions for possible 2-homogeneous limits. This is the content of the following 2 lemmas.

Lemma 7.14. *Suppose that for some subsequence,* $\tilde{w}_{r_{k_l}} \rightharpoonup q$ *in* $W^{1,2}(B_1)$ *and* $\frac{r_{k_l}^{2\gamma}}{H(r_{k_l},w)} \downarrow 0$ *as* $r_{k_l} \downarrow 0$. *Then*

$$\int_{\partial B_1} q(p - p_0) \ge 0 \;\; \forall p \in P. \tag{7.10}$$

Proof. Define $h_r := \|w_r\|_{L^2(\partial B_1)}$ and $\varepsilon_r = \frac{h_r}{r^2}$ and note that by the compactness of the trace operator $\tilde{w}_{r_{k_l}} \to q$ in $L^2(\partial B_1)$. Then for any $p \in P$ by Monneau monotonicity applied to u and $f(0)p$,

$$\int_{\partial B_1} \left(\frac{w_r}{r^2} + f(0)p_0 - f(0)p \right)^2 + Cr^\alpha = \int_{\partial B_1} \left(\frac{u(rx) - f(0)p(rx)}{r^2} \right)^2 + Cr^\alpha$$
$$\downarrow \int_{\partial B_1} (f(0)p_0 - f(0)p)^2.$$

Hence we obtain that for all $r > 0$ and for all $p \in P$

$$\int_{\partial B_1} \left(\frac{w_r}{r^2} + f(0)p_0 - f(0)p \right)^2 + Cr^\alpha \geq \int_{\partial B_1} (f(0)p_0 - f(0)p)^2.$$

Expanding the squares and taking $r = r_{k_l}$ we obtain

$$C\frac{r_{k_l}^\alpha}{\varepsilon_{r_{k_l}}} + \varepsilon_{r_{k_l}} \int_{\partial B_1} \tilde{w}_{r_{k_l}}^2 + 2\int_{\partial B_1} \tilde{w}_{r_{k_l}} (f(0)p_0 - f(0)p) \geq 0.$$

Observing that

$$\frac{r_{k_l}^\alpha}{\varepsilon_{r_{k_l}}} = \frac{r_{k_l}^{2+\alpha}}{(H(r_{k_l}, w))^{1/2}}$$

allows us to recover the claim after taking the limit $l \to \infty$. □

The following is Lemma 2.12 from [10].

Lemma 7.15. *Let $p_0 \in P$ and $q \not\equiv 0$ be a 2 homogeneous harmonic polynomial satisfying (7.10). Then in some appropriate system of co-ordinates, we have that*

$$p_0(x) = \frac{1}{2} \sum_{i=m+1}^{n} \mu_i x_i^2, \quad q(x) = \nu \sum_{i=m+1}^{n} x_i^2 - \sum_{j=1}^{m} \nu_j x_j^2, \tag{7.11}$$

where $\mu_i, \nu > 0$, $\sum_{i=m+1}^n \mu_i = 1$, $(n-m)\nu = \sum_{j=1}^m \nu_j$ and $|\nu_j| \leq \nu$ for all $j = 1, \ldots, m$.

Proof. See Lemma 2.12 from [10] with the small correction made in [9]. □

We will require one more lemma which establishes the boundedness of w_r in some appropriate Hólder space in the case that $m = n - 1$.

Lemma 7.16. *Suppose $m = n - 1$ and with no loss of generality $p_0(x) = \frac{1}{2}x_n^2$. For some $\beta > 0$ and $C = C(n, \alpha, \|f\|_{C^{0,\alpha}})$ we have that*

$$\|w_r\|_{C^{0,\beta}(B_{1/2})} \leq C(\|w_r\|_{L^2(B_1)} + r^{2+\alpha})$$

Proof. The proof will be completed in several steps.

Step 1: We have the L^∞ estimate

$$\|w_r\|_{L^\infty(B_1)} \le C(\|w_r\|_{L^2(B_1)} + r^{2+\alpha}).$$

First note that for C large enough (depending on $[f]_{C^{0,\alpha}}$) we have that $\Delta\left(w_r + Cr^{2+\alpha}\frac{|x|^2}{2n}\right) \ge 0$ in $\{u_r > 0\}$ while $w_r + Cr^{2+\alpha}\frac{|x|^2}{2n} \le Cr^{2+\alpha}\frac{|x|^2}{2n}$ in $\{u_r = 0\}$. As a consequence we have that $\max\{w_r + Cr^{2+\alpha}\frac{|x|^2}{2n}, Cr^{2+\alpha}\frac{|x|^2}{2n}\}$ is subharmonic. Moreover, since $\max\{w_r + Cr^{2+\alpha}\frac{|x|^2}{2n}, Cr^{2+\alpha}\frac{|x|^2}{2n}\} = \max\{w_r, 0\} + Cr^{2+\alpha}\frac{|x|^2}{2n}$, we have that $(w_r)_+ + Cr^{2+\alpha}\frac{|x|^2}{2n}$ is subharmonic. On the other hand, for C large enough (also depending on $[f]_{C^{0,\alpha}}$), we have that $\Delta\left(-w_r + Cr^{2+\alpha}\frac{|x|^2}{2n}\right) \ge 0$ everywhere and so is subharmonic and hence $(-w_r + Cr^{2+\alpha}\frac{|x|^2}{2n})_+$ is also subharmonic.

So for any $x \in B_{1/2}$ we have by the mean value property of subharmonic functions

$$\begin{aligned}
|w_r|(x) &= (w_r)_+(x) + (w_r)_-(x) \\
&\le (w_r)_+(x) + Cr^{2+\alpha}\frac{|x|^2}{2n} + (-w_r(x) + Cr^{2+\alpha}\frac{|x|^2}{2n})_+ \\
&\le \fint_{B_{1/2}(x)} (w_r)_+(y) + Cr^{2+\alpha}\frac{|y|^2}{2n}dy + \fint_{B_{1/2}(x)} (-w_r(y) + Cr^{2+\alpha}\frac{|y|^2}{2n})_+ dy \\
&\le C(\|w_r\|_{L^2(B_1)} + Cr^{2+\alpha}),
\end{aligned}$$

and so the L^∞ estimate follows.

Step 2: We have the L^2 estimate

$$\int_{B_{1/2}} |\nabla w_r|^2 \le C(\|w_r\|_{L^2(B_1)} + r^{2+\alpha}).$$

Indeed choosing $\eta \in C_c^\infty(B_1)$ with $\eta \equiv 1$ on $B_{1/2}$ and $|\nabla\eta| \le 4$ we have that

$$\begin{aligned}
\int_{B_1} \eta^2|\nabla w_r|^2 &= -\int_{B_1} w_r\nabla\cdot(\eta^2\nabla w_r) \\
&= -2\int_{B_1} w_r\eta\nabla\eta\cdot\nabla w_r - w_r\Delta w_r\eta^2 \\
&\le -2\int_{B_1} w_r\eta\nabla\eta\cdot\nabla w_r + Cr^{2+\alpha}\int_{B_1}|w_r| - r^4 f(0)\int_{B_1} pf_r \\
&\le 2\|\eta\nabla w_r\|_{L^2(B_1)}\|w_r\|_{L^2(B_1)} + Cr^{2+\alpha}\|w_r\|_{L^2(B_1)} \\
&= 2\|w_r\|_{L^2(B_1)}(\|\eta\nabla w_r\|_{L^2(B_1)} + Cr^{2+\alpha}),
\end{aligned}$$

and after applying Young's inequality we achieve the announced estimate.

Step 3: We now show we can control the $C^{0,\alpha}$ semi-norm in the directions e_j for $j \neq n$. For $j = 1, \dots, n-1$ and any $t \in (0,1)$ we define

$$(\delta_j^{\pm} w_r)(x) = \frac{w_r(x \pm te_j) - w_r(x)}{t^\alpha}$$

and observe the symmetry $(\delta_j^{\pm} w_r)(x \mp te_j) = -(\delta_j^{\mp} w_r)(x)$. Moreover, by using a Taylor expansion with remainder we obtain $\left\|\delta_j^{\pm} w_r\right\|_{L^2(B_{1/2})} \leq Ct^{1-\alpha}\|\nabla w_r\|_{L^2(B_1)}$ and so by step 2 we have that

$$\left\|\delta_j^{\pm} w_r\right\|_{L^2(B_{1/2})} \leq C(\|w_r\|_{L^2(B_1)} + r^{2+\alpha}) \tag{7.12}$$

uniformly in t. Since p_0 is constant in every e_j direction for $j \neq n$ we have that $(\delta_j^{\pm} w_r)(x) = (\delta_j^{\pm} u_r)(x)$ and so in $\{u_r > 0\} \cap B_1$ we have that $\Delta(\delta_j^{\pm} w_r) \leq Cr^{2+\alpha}$ while in $\{u_r = 0\} \cap B_1$ we have that $\delta_j^{\pm} w_r \geq 0$. Consequently, $H(x) := \min\left(\delta_j^{\pm} w_r + Cr^{2+\alpha}\frac{1-|x|^2}{2n}, 0\right)$ is super harmonic and by the minimum principle there exists some $z \in \partial B_{1/2}$ so that

$$\min_{B_{1/2}} H(x) = H(z) \geq \fint_{B_{1/4}(z)} H(x)dx.$$

On the other hand we have that

$$\left|\fint_{B_{1/4}(z)} H(x)dx\right| \leq C(n)\int_{B_{3/4}} |H(x)|dx \leq C(n)\int_{B_{3/4}} \delta_j^{\pm} w_r + Cr^{2+\alpha}\frac{1-|x|^2}{2n},$$

and so using (7.12) we obtain

$$\min_{B_{1/2}} \delta_j^{\pm} w_r \geq -C(\|w_r\|_{L^2(B_1)} + r^{2+\alpha}).$$

Applying the symmetry property yields

$$\max_{B_{1/2}} \delta_j^{\pm} w_r \leq C(\|w_r\|_{L^2(B_1)} + r^{2+\alpha}).$$

We can now conclude using Lemma C.1 from [11] which states that having the C^α control on w_r in the e_j directions for $j = 1, \dots, n-1$ and the L^2 control on $\partial_n w_r$ gives that w_r is bounded in the $C^\beta(B_1)$ semi-norm for some $\beta > 0$ with the estimate

$$[w_r]_{C^\beta(B_{1/2})} \leq C(\|w_r\|_{L^2(B_1)} + r^{2+\alpha}).$$

□

We can now give the

Proof of Proposition 7.13. Step 1: We show that $(\tilde{w}_r)_{r>0}$ is uniformly bounded in $W^{1,2}(B_1)$. Since the renormalisation yields

$$\int_{\partial B_1} \tilde{w}_r{}^2 = 1,$$

using the truncated frequency we can bound $D(1, \tilde{w}_r)$ as

$$D(1, \tilde{w}_r) \leq 2\phi^\gamma(r, \tilde{w}) \leq 2\phi^\gamma(1, \tilde{w}) \leq C. \tag{7.13}$$

This gives the uniform bounds $\|\tilde{w}_r\|_{W^{1,2}(B_1)} \leq C$ for all $0 < r < 1$ and hence given any sequence $r_k \downarrow 0$ there exists a subsequence $r_{k_l} \downarrow 0$ such that

$$\tilde{w}_{r_{k_l}} \rightharpoonup q \quad \text{in} \quad W^{1,2}(B_1),$$

and hence

$$\tilde{w}_{r_{k_l}} \to q \quad \text{in} \quad L^2(B_1).$$

For later reference we also note that these same uniform bounds and hence convergence can also be achieved in B_2 (or any B_R for that matter, but B_2 is all we will use). Indeed, just by scaling in (7.13) with $r < \frac{1}{2}$ we obtain that $D(2, \tilde{w}_r) \leq C$. Then $H(2, \tilde{w}_r)$ is controlled using Lemma 7.8 again with $r < \frac{1}{2}$.

Step 2: We prove (a) and so suppose that $m = n - 1$. The case a(i) is handled in Remark 7.9 and so we assume that a(i) fails. By the discussion in Remark 7.9 we have that $\frac{r^{2\gamma}}{H(r,w)} \downarrow 0$ as $r \downarrow 0$ and $\lambda_* = \phi(0^+, w) = \phi^\gamma(0^+, w)$. Observe that Δq is a non-positive measure supported on L. Indeed there holds for some $C > 0$ large enough (that depends on $[f]_{C^{0,\alpha}}$) that

$$\Delta(w_r - Cr^{2+\alpha}|x|^2) = -2nCr^{2+\alpha} + r^2(f(rx) - f(0)) - r^2 f(0)\chi_{\{u_r=0\}} \leq 0.$$

As a consequence Δw_r has almost a sign so that we can compute, using integration by parts with some $\eta \in C_c^\infty(B_2)$ that satisfies $\eta \equiv 1$ on B_1, that

$$\begin{aligned}\int_{B_1} \left|\Delta(w_r - Cr^{2+\alpha}|x|^2)\right| &= \int_{B_1} -\Delta(w_r - Cr^{2+\alpha}|x|^2) \\ &\leq \int_{B_2} -\eta\Delta(w_r - Cr^{2+\alpha}|x|^2) \\ &\leq C(\|w_r\|_{L^2(B_2)} + r^{2+\alpha}).\end{aligned}$$

Re-arranging this and dividing by $\|w_r\|_{L^2(\partial B_1)}$ we obtain the L^1 bound

$$\int_{B_1} |\Delta\tilde{w}_r| \leq C\left(\|\tilde{w}_r\|_{L^2(B_2)} + \frac{r^{2+\alpha}}{\|w_r\|_{L^2(\partial B_1)}}\right),$$

so that $\Delta\tilde{w}_{r_{k_l}}$ converges weakly-* as measures to Δq (up to extracting a further subsequence). Moreover there holds that

$$\frac{1}{r_{k_l}^2}\Delta\tilde{w}_{r_{k_l}} = \begin{cases} -f(0) & \text{in} \quad \{u_{r_{k_l}} = 0\} \\ O(r_{k_l}^\alpha) & \text{in} \quad \{u_{r_{k_l}} > 0\} \end{cases}$$

as well as the convergence of the sets $\{u_r = 0\}$ to a subset of L. It follows that Δq must be supported on L and satisfies $\Delta q \le 0$. By Lemma 7.16 we have that (up to extracting a subsequence) $\tilde{w}_{r_{k_l}} \to q$ in $C^0_{loc}(B_1)$ and so we have that $q \ge 0$ on L and

$$\lim_{l\to\infty} \int_{B_1} \tilde{w}_{r_{k_l}} \Delta \tilde{w}_{r_{k_l}} = \int_{B_1} q\Delta q.$$

On the other hand we also have by (7.7) that

$$\int_{B_1} \tilde{w}_r \Delta \tilde{w}_r \ge -Cr^{\varepsilon}$$

and so it must hold that $\int_{B_1} q\Delta q \ge 0$. Since $\Delta q \le 0$ is supported on L and $q \ge 0$ on L, it follows that $q\Delta q \le 0$ and so we conclude that $q\Delta q \equiv 0$ in B_1. This shows that q solves (7.9) in B_1.
Another consequence of the locally uniform convergence is that $\tilde{w}_{r_{k_l}} \to q$ in $W^{1,2}_{loc}(B_1)$. Indeed, let $\eta \in C^\infty_c(B_1)$ and note that using integration by parts

$$\begin{aligned}
\lim_{l\to\infty} \int_{B_1} \left|\nabla(\eta \tilde{w}_{r_{k_l}})\right|^2 &= \lim_{l\to\infty} \left(- \int_{B_1} \eta \tilde{w}^2_{r_{k_l}} \Delta\eta + 2\eta \tilde{w}_{r_{k_l}} \nabla \tilde{w}_{r_{k_l}} \cdot \nabla\eta + \eta^2 \tilde{w}_{r_{k_l}} \Delta \tilde{w}_{r_{k_l}} \right) \\
&= - \int_{B_1} \eta q^2 \Delta\eta + 2\eta q \nabla q \cdot \nabla\eta + \eta^2 q\Delta q \\
&= \int_{B_1} |\nabla(\eta q)|^2.
\end{aligned}$$

Now, by the strong convergence in $W^{1,2}_{loc}(B_1)$ we have for any $R > 0$

$$\phi(R,q) = \lim_{l\to\infty} \phi(R, \tilde{w}_{r_{k_l}}) = \lim_{l\to\infty} \phi(Rr_{k_l}, w) = \lim_{l\to\infty} \phi^{\gamma}(Rr_{k_l}, w) = \lambda_*,$$

so that q is λ_* homogenous. By homogeneity we can extend q to all of $\mathbb{R}^n$ and so q satisfies (7.9).

Finally, we will show that $\lambda_* \ge 2 + \alpha_0$ for some dimensional constant α_0. First note that any blow-up q must satisfy (7.9), (7.14), $q(0) = 0$ and by the compactness of the trace operator we also have that $\|q\|_{L^2(\partial B_1)} = 1$. Suppose there exists a sequence of functions $q^{(k)}$ satisfying these conditions with $\lambda^k_* \downarrow 2$. Then there is a limiting function $q^{(\infty)}$ with $\lambda^{(\infty)}_* = 2$ satisfying these assumptions and hence a 2 homogeneous solution to the thin obstacle problem. Since these are classified, $q^{(\infty)}$ is a quadratic harmonic polynomial. Applying Lemma 7.5 we have that

$$p_0(x) = \frac{f(0)}{2} x_n^2$$

and

$$q^{(\infty)}(x) = \nu x_n^2 - \sum_{j=1}^{n-1} \nu_j x_j^2.$$

However, since $q^{(\infty)} \geq 0$ on L, we reach that $\nu = \sum_{j=1}^{n-1} \nu_j \leq 0$ contradicting the fact that $\nu > 0$.

Step 3: We prove (b) and so suppose that $m \leq n-2$ and that $b(i)$ fails.
Carrying on from step 1 we know that $\Delta q \leq 0$ and is supported on L. However, in this case L has codimension at least 2 and $q \in W^{1,2}(B_1)$ and so we must have that $\Delta q = 0$.
To show that q is homogeneous we will first show that in this case we still have the strong $W^{1,2}_{loc}(B_1)$ convergence. Observe that $\int_{B_1} q\Delta q = 0$ since $\Delta q = 0$ and by (7.7) we have that $\limsup_{r\downarrow 0} \int_{B_1} \tilde{w}_{r_{k_l}} \Delta \tilde{w}_{r_{k_l}} \geq 0$. Then with $\eta \in C_c^\infty(B_1)$ we can integrate by parts as before and obtain

$$\begin{aligned}
\limsup_{l\to\infty} \int_{B_1} \left|\nabla(\eta \tilde{w}_{r_{k_l}})\right|^2 &= \limsup_{l\to\infty} \left(-\int_{B_1} \eta \tilde{w}_{r_{k_l}}^2 \Delta\eta + 2\eta \tilde{w}_{r_{k_l}} \nabla \tilde{w}_{r_{k_l}} \cdot \nabla\eta + \eta^2 \tilde{w}_{r_{k_l}} \Delta \tilde{w}_{r_{k_l}} \right) \\
&\leq \limsup_{l\to\infty} \left(-\int_{B_1} \eta \tilde{w}_{r_{k_l}}^2 \Delta\eta + 2\eta \tilde{w}_{r_{k_l}} \nabla \tilde{w}_{r_{k_l}} \cdot \nabla\eta \right) \\
&= -\int_{B_1} \eta q^2 \Delta\eta + 2\eta q \nabla q \cdot \nabla\eta + \eta^2 q\Delta q \\
&= \int_{B_1} |\nabla(\eta q)|^2.
\end{aligned}$$

By weak sequential lower semicontinuity of the norm we also have that

$$\int_{B_1} |\nabla(\eta q)|^2 \leq \liminf_{l\to\infty} \left|\nabla(\eta \tilde{w}_{r_{k_l}})\right|^2,$$

and so the convergence holds strongly in $W^{1,2}_{loc}$. Arguing as in Step 2 we conclude that q is λ_* homogeneous and since $\lambda_* < \gamma$ and q is harmonic it follows that $\lambda_* = 2$. That is, q is a 2 homogeneous harmonic polynomial as claimed. □

7.5 Improving the covering manifold

We begin this section by first proving the analogue of Lemma 3.11 from [10] suitable for our purposes (cf. Proposition 7.7).

Proposition 7.17. *Let $\lambda = l + \beta$ for some $l \in \mathbb{N}$, $0 < \beta \leq 1$ and let $C > 0$ and define the set*

$$S_{m,\lambda,C} = \{x_0 \in \Sigma_m : H(r, u - f(x_0)p_{x_0}) \leq Cr^\lambda \ \forall r \in (0, 1/2)\}.$$

Then $S_{m,\lambda,C}$ is locally contained in a $C^{l-1,\beta}$ manifold of dimension m.

The proof is the same as that of Lemma 3.11 from [10] and will need the following version of the Whitney extension theorem.

Theorem 7.18 (Whitney's Extension Theorem)**.** *Let $\beta \in (0,1]$, $l \in \mathbb{N}$, E a compact subset of $\mathbb{R}^n$ and $h : E \to \mathbb{R}$ a map. Suppose that for any $x_0 \in E$ there exists a polynomial P_{x_0} of degree l such that*

- $P_{x_0}(x_0) = h(x_0)$
- $|D^k P_{x_0}(x) - D^k P_x(x)| \le C|x - x_0|^{l+\beta-k}$ *for all* $x \in E$ *and* $k = 0, \cdots, l$, *where* $C > 0$ *is independent of* x_0.

Then h *extends to a* $C^{l,\beta}$ *function on* $\mathbb{R}^n$ *such that*

$$f(x) = P_{x_0}(x) + O(|x - x_0|^{l+\beta}),$$

for all $x_0 \in E$.

Proof of Proposition 7.17. We first define the set

$$S_{\lambda,C} = \{x_0 \in \Sigma : \ H(r, u - f(0)p_{x_0}) \le Cr^\lambda, \ r \in (0, 1/2)\},$$

and note that this set is closed and $S_{m,\lambda,C} \subset S_{\lambda,C}$.

Now define the compact set $E = S_{\lambda,C} \cap \overline{B_{1/4}}$ and let $h : E \to \mathbb{R}$ be identically 0. Just as in the proof of Theorem 7.5 we will define $P_{x_0}(x) = f(x_0)p_{x_0}(x - x_0)$ and note that the first condition is trivially satisfied, that is $P_{x_0}(x_0) = f(x_0)p_{x_0}(0) = 0$.

Now let $x_0, x \in E$ and let $\rho = |x - x_0|$. With no loss of generality we will assume $x_0 = 0$. Moreover we note that if $w = u - f(0)p_0$ we have that

$$\begin{aligned}
\int_{B_1} w^2(\rho\cdot) &= \frac{1}{\rho^n}\int_{B_\rho} w^2 \\
&= \frac{1}{\rho^n}\int_0^\rho r^{n-1}\int_{\partial B_1} w^2(r\theta)d\sigma(\theta)dr \\
&= \frac{1}{\rho^n}\int_0^\rho r^{n-1}H(r, w)dr \\
&\le \frac{1}{\rho^n}\int_0^\rho r^{2\lambda+n-1}dr \\
&\le C\rho^{2\lambda}
\end{aligned}$$

and so we have the same control over the squared $L^2(B_1)$ norm of w^2.

Now since $\left|\frac{x}{\rho}\right| = 1$ we have that $B_1(0) \subset B_2(\frac{x}{\rho})$ and so

$$\begin{aligned}
\|(P_0 - P_x)(\rho\cdot)\|_{L^2(B_1)} &\le \|u(\rho\cdot) - P_0(\rho\cdot)\|_{L^2(B_1)} + \|u(\rho\cdot) - P_x(\rho\cdot)\|_{L^2(B_1)} \\
&= \|u(\rho\cdot) - f(0)p_0(\rho\cdot)\|_{L^2(B_1)} + \left\|u(\rho\cdot) - f(x)p_x(\rho(\cdot - \frac{x}{\rho}))\right\|_{L^2(B_1)} \\
&\le \|u(\rho\cdot) - f(0)p_0(\rho\cdot)\|_{L^2(B_1)} + \left\|u(\rho\cdot) - f(x)p_x(\rho(\cdot - \frac{x}{\rho}))\right\|_{L^2(B_2(\frac{x}{\rho}))} \\
&\le \|u(\rho\cdot) - f(0)p_0(\rho\cdot)\|_{L^2(B_1)} + \|u(x + \rho\cdot) - f(x)p_x(\rho\cdot)\|_{L^2(B_2)} \\
&\le C\rho^\lambda.
\end{aligned}$$

Now, using the equivalence of the $L^2(B_1)$ norm with the $C^l(B_1)$ norm on the space of quadratic polynomials we deduce the existence of a constant $C > 0$ such that

$$\left|D^k P_{x_0}(x) - D^k P_x(x)\right| \le C|x - x_0|^{\lambda - k}$$

for all $x \in E$ and $k = 0, \cdots, l$. Then by Theorem 7.18 h extends to a $C^{l,\beta}(\mathbb{R}^n)$ function F and the proof continues in the exact same way as that of Theorem 7.5. □

We can now finally improve the covering manifold of the individual strata combining Proposition 7.17 with the blow-up analysis Proposition 7.13 and Corollary 7.12. We begin with the top dimensional stratum, Σ_{n-1}.

Theorem 7.19. *Assuming the same notation as in Proposition 7.13 and Proposition 7.17 we have that $\Sigma_{n-1} = S_{n-1,2+\alpha_0,C}$ where $C > 0$ is a constant depending only on n, $\|f\|_{C^{0,\alpha}}$ and $\|u\|_{C^{1,1}}$. In particular, Σ_{n-1} can be covered by a C^{1,α_0} manifold of dimension m.*

Proof. Suppose x_0 is a point in Σ_{n-1}, by the blow up analysis either $\phi^\gamma(0^+, u(x_0 + \cdot) - f(x_0)p_{x_0}) = \gamma$ or $\phi^\gamma(0^+, u(x_0 + \cdot) - f(x_0)p_{x_0}) \ge 2 + \alpha_0$ where $2 + \alpha_0 < \gamma$. In both cases Corollary 7.12 gives that $H^\gamma_{2+\alpha_0}(r, u(x_0 + \cdot) - f(x_0)p_{x_0})$ is almost monotone so that for all $r \in (0, 1/2)$ we have that

$$r^{-2(2+\alpha_0)} H(r, u(x_0 + \cdot) - f(x_0)p_{x_0}) \le H^\gamma_{2+\alpha_0}(1/2, u(x_0 + \cdot) - f(x_0)p_{x_0}) + C\left(\frac{1}{2}\right)^\varepsilon \le C$$

where C is a constant depending only on n, $\|f\|_{C^{0,\alpha}}$ and $\|u\|_{C^{1,1}}$, in particular independent of x_0. This proves $\Sigma_{n-1} = S_{n-1,2+\alpha_0,C}$ and so applying Proposition 7.17 we achieve the result. □

On the lower dimensional strata we cannot improve the covering of the entire stratum Σ_m using Theorem 7.17 since in this case we can have second order blow-ups with frequency 2. This motivates defining the set of anomalous points

$$\Sigma^a_m = \{x_0 \in \Sigma_m; \quad \phi(0^+, u(x_0 + \cdot) - f(x_0)p_{x_0}) = 2\}.$$

Then the set of 'good' points $\Sigma^g_m = \Sigma_m \backslash \Sigma^a_m$ can be covered using Proposition 7.17 with a $C^{1,\frac{\alpha}{8}}$ manifold of dimension m since every point in Σ^g_m has $\phi^\gamma(0^+, u(x_0 + \cdot) - f(x_0)p_{x_0}) = \gamma$.

However, we can still improve the covering of the entire stratum Σ_m from an abstract modulus of continuity (Theorem 7.5) to a quantitative modulus of continuity, which is the result achieved in [5]. We provide the proof as given in [10].

Theorem 7.20. *Σ_m can be locally covered by a $C^{1,\log^{\varepsilon_0}}$ m dimensional manifold for some dimensional constant ε_0.*

Proof. As noted above, the only problematic set that we need to cover is Σ_m^a as the all other points can be covered using Proposition 7.17 with higher regularity. To this end we just need to show that for some $\varepsilon_0 > 0$

$$H(1, u(x_0 + r\cdot) - f(x_0)p_{x_0}(r\cdot))^{1/2} \le Cr^2 \log^{-\varepsilon_0}(1/r) \ \forall x_0 \in \Sigma^a \cap B_{1/2}, \ \forall r \in (0, 1/2), \tag{7.14}$$

and the appropriate version of the Whitney extension theorem coupled with the exact same argument as in the proof of Proposition 7.17 will yield the result.

We first recall Caffarelli's semi-convexity estimate (see [2, Theorem 1]), which states that for some dimensional constants ε_0 and C and any direction $e \in \mathbb{S}^{n-1}$ that

$$\partial_{ee} u(x) \ge -C \log^{-\varepsilon_0}(1/|x|) \text{ in } B_{1/2}.$$

For simplicity of notation we will show (7.14) at $x_0 = 0$ and so suppose $0 \in \Sigma_m^a$ and define $L = \{p_0 = 0\}$.

Now suppose that (7.14) did not hold with this ε_0, that is, there exits some sequence $r_k \downarrow 0$ such that for all $M > 0$

$$H(r, u(r\cdot) - f(0)p_0(r\cdot))^{1/2} \ge M r_k^2 \log^{-\varepsilon_0}(1/r_k).$$

Then for any $e \in \mathbb{S}^{n-1} \cap L$

$$\begin{aligned}
\partial_{ee}\tilde{w}_{r_k} &= \partial_{ee}\left(\frac{u(r_k\cdot) - f(0)p_0(r_k\cdot)}{(H(1, u(_k\cdot) - f(0)p_0(r_k\cdot)))^{1/2}}\right) \\
&= \frac{r_k^2 \partial_{ee} u(r_k\cdot)}{(H(1, u(_k\cdot) - f(0)p_0(r_k\cdot)))^{1/2}} \\
&\ge -C \frac{r_k^2 \log^{-\varepsilon_0}(1/r_k)}{(H(1, u(_k\cdot) - f(0)p_0(r_k\cdot)))^{1/2}} \\
&\ge -\frac{C}{M}.
\end{aligned}$$

Now by Proposition 7.13(b) we have that there exists a subsequence r_{k_l} such that $\tilde{w}_{r_{k_l}} \to q$ in L^2. (Note that we are already in case b(ii) by assumption, or else we would have the much stronger estimate in b(i)). Consequently, we have that

$$\partial_{ee} q \ge -\frac{C}{M} \text{ for all } e \in \mathbb{S}^{n-1} \cap L. \tag{7.15}$$

On the other hand, q is a 2-homogenous harmonic polynomial satisfying (7.10) and so we have by Lemma 7.15 and the compactness of the trace operator that

$$D^2 q|_L \le 0, \ D^2 q|_{L^\perp} \ge 0, \ tr(D^2 q) = 0 \text{ and } \|q\|_{L^2(\partial B_1)}. \tag{7.16}$$

This implies that there exists some $e' \in \mathbb{S}^{n-1} \cap L$ such that

$$\partial_{e'e'} \leq -c_1 < 0 \tag{7.17}$$

for some dimensional constant c_1. Indeed, suppose that (7.17) did not hold so that there existed a sequence $q^{(k)} \to q^{(\infty)}$ where every $q^{(k)}$ satisfies (7.16) as well as for every $k > 0$

$$\partial_{ee} q^{(k)} \geq -\frac{1}{k}$$

for every $e \in \mathbb{S}^{n-1} \cap L$. Then this would imply that $D^2 q^{(\infty)}|_L \geq 0$ and in light of (7.16) then $D^2 q^{(\infty)}|_L = 0$. At the same time (7.16) states that $D^2 q|_{L^\perp} \geq 0$ and $tr(D^2 q) = 0$ and so it must follow that $q \equiv 0$, a contradiction. This establishes (7.17) which in turn contradicts (7.15) for M large enough, consequently establishing (7.14). □

7.6 Dimension Reduction Argument

We will now investigate how big Σ_m^a can be. For this we will need to establish some properties of Hausdorff measures and dimension.

7.6.1 Hausdorff Measure and Dimension

In this subsection we are following the treatment in [6]. Given any $\beta \in \mathbb{N}$ and $\delta > 0$ we define the Hausdorff premeasure of a set E as

$$\mathcal{H}_\delta^\beta(E) = \inf\left\{\sum_{i=1}^{\infty} \operatorname{diam}(E_j)^\beta : \ E \subset \bigcup_{j=1}^{\infty} E_j, \ \operatorname{diam}(E_j) < \delta\right\}.$$

We then define the β dimensional Hausdorff measure as

$$\mathcal{H}^\beta(E) = \lim_{\delta \downarrow 0} \mathcal{H}_\delta^\beta(E) = \sup_{\delta > 0} \mathcal{H}_\delta^\beta(E).$$

Finally we can define the Hausdorff dimension of a set E as

$$\dim_{\mathcal{H}}(E) = \inf\left\{\beta > 0 : \ \mathcal{H}^\beta(E) = 0\right\}.$$

We have the following equivalent characterisation of $\dim_{\mathcal{H}}$.

Proposition 7.21. *For any $\delta > 0$*

$$\dim_{\mathcal{H}}(E) = \inf\left\{\beta > 0 : \ \mathcal{H}_\delta^\beta(E) = 0\right\}.$$

Proof. Clearly if $\sup_{\delta>0}\mathcal{H}^\beta_\delta(E)=0$ then $\mathcal{H}^\beta_\delta(E)\le 0$ and so $\mathcal{H}^\beta_\delta(E)=0$. On the other hand if $\mathcal{H}^\beta_\delta(E)=0$ for any $\delta>0$ we have that for any $\varepsilon>0$ there exists a collection $\{E_j\}_{j\in\mathbb{N}}$ such that

$$\sum_{j\in\mathbb{N}}(\mathrm{diam}(E_j))^\beta<\varepsilon,$$

and so for all $j\in\mathbb{N}$ we have that $\mathrm{diam}(E_j)\le\varepsilon^{1/\beta}$. Hence we have that $\mathcal{H}^\beta_{\varepsilon^{1/\beta}}(E)<\varepsilon$ and so as $\varepsilon\downarrow 0$ we obtain that $\mathcal{H}^\beta(E)=0$. □

Given some set E with $\mathcal{H}^\beta(E)>0$ we will call $x\in E$ a **density point** if

$$\limsup_{r\downarrow 0}\frac{\mathcal{H}^\beta(B_r(x)\cap E)}{r^\beta}\ge 1.$$

Note that if $\mathcal{H}^\beta(E)=0$ then there would exist no density points and so $\mathcal{H}^\beta_\infty(E)>0$ is a necessary assumption. Moreover, we have the following result.

Proposition 7.22. *Given a set $E\subset\mathbb{R}^n$ and some $\beta\in(0,n]$ such that $0<\mathcal{H}^\beta(E)<\infty$, we have that $\mathcal{H}^\beta$-almost every point in E is a density point.*

Proof. Given any $\delta>0$ and $\tau\in(0,1)$ we define the set

$$E(\delta,\tau)=\left\{x\in E:\ \mathcal{H}^\beta_\delta(E\cap C)\le\tau(\mathrm{diam}(C))^\beta \text{ for all } C\subset\mathbb{R}^n \text{ with } x\in C \text{ and } \mathrm{diam}(C)\le\delta\right\}$$

Observe that $\mathcal{H}^\beta_\delta(E(\delta,\tau))=0$. Indeed, for some $\varepsilon>0$ let $\{E_j\}_{j\in\mathbb{N}}$ be a collection of subsets so that $E\subset\cup_j E_j$, $\mathrm{diam}(E_j)\le\delta$ and $\sum(\mathrm{diam}(E_J))^\beta\le\mathcal{H}^\beta_\delta(E(\delta,\tau))+\varepsilon$. Moreover, impose that $E_j\cap E(\delta,\tau)\neq\varnothing$. Then we have that

$$\begin{aligned}\mathcal{H}^\beta_\delta(E(\delta,\tau))&\le\sum_j\mathcal{H}^\beta_\delta(E_j\cap E(\delta,\tau))\\&\le\sum_j\mathcal{H}^\beta_\delta(E_j\cap E)\\&\le\tau\sum_j\mathrm{diam}(E_j)^\beta\\&\le\tau\mathcal{H}^\beta_\delta(E(\delta,\tau))+\varepsilon.\end{aligned}$$

Consequently we have that $\mathcal{H}^\beta_\delta(E(\delta,\tau))\le\tau\mathcal{H}^\beta_\delta(E(\delta,\tau))$ and so since $\tau\in(0,1)$ and $\mathcal{H}^\beta_\delta(E(\delta,\tau))\le\mathcal{H}^\beta_\delta(E)\le\mathcal{H}^\beta(E)<\infty$ we conclude that $\mathcal{H}^\beta_\delta(E)=0$.

Now suppose that $x\in E$ and

$$\limsup_{r\downarrow 0}\frac{\mathcal{H}^\beta(B_r(x)\cap E)}{r^\beta}<1.$$

Then there exists a $\delta>0$ such that

$$\frac{\mathcal{H}^\beta(B_r(x)\cap E)}{r^\beta}<1-\delta,\ \ \forall r\in(0,\delta].$$

Now let $C \subset \mathbb{R}^n$ with $\operatorname{diam}(C) \leq \delta$ and $x \in C$. Then we have since $\operatorname{diam}(C) \leq \delta$

$$\begin{aligned}\mathcal{H}^\beta_\delta(C \cap E) &= \mathcal{H}^\beta_\infty(C \cap E)\\ &\leq \mathcal{H}^\beta_\infty(B_{\operatorname{diam}(C)}(x) \cap E)\\ &\leq (1-\delta)(\operatorname{diam}(C))^\beta.\end{aligned}$$

This shows that $x \in E(\delta, 1-\delta)$ and so

$$\left\{x \in E: \ \limsup_{r \downarrow 0} \frac{\mathcal{H}^\beta(B_r(x) \cap E)}{r^\beta} < 1\right\} \subset \bigcup_{k \in \mathbb{N}} E(\frac{1}{k}, 1 - \frac{1}{k}).$$

This concludes the proof $\square$

Remark 7.23. *Since we dropped all re-normalisation constants this constant* 1 *is not so standard, it should be* $\frac{1}{2^\beta}$*. Moreover, the above definition of density point as well as the proof also works if I replace* $\mathcal{H}^\beta$ *with* $\mathcal{H}^\beta_\infty$*. In light of Proposition 7.21 we can also use* $\mathcal{H}^\beta_\infty$ *to characterise* $\dim_{\mathcal{H}}$ *and so from here on we will exclusively work with the premeasure* $\mathcal{H}^\beta_\infty$*.*

For what follows we will assume that $0 < \mathcal{H}^\beta_\infty(E) < \infty$ and that we are centred on $0 \in E$ where 0 is a density point. Explicitly, this means there exists some sequence $r_k \downarrow 0$ such that

$$\lim_{r_k \downarrow 0} \frac{\mathcal{H}^\beta_\infty(B_{r_k} \cap E)}{r_k^\beta} \geq 1$$

We now define the **accumulation set** for E along r_k at 0 as

$$\mathcal{A} = \mathcal{A}_{E,\{r_k\}} = \left\{z \in \overline{B}_{1/2} : \ \exists\ (z_l)_{l \in \mathbb{N}},\ (k_l)_{l\in\mathbb{N}} \text{ such that } z_l \in r_{k_l}^{-1} E \cap B_{1/2} \text{ and } z_l \to z\right\}$$

Proposition 7.24. *If* $0 < \mathcal{H}^\beta_\infty(E) < \infty$ *then* $\mathcal{H}^\beta_\infty(\mathcal{A}) > 0$*.*

Proof. First note that for k large enough we have that

$$\mathcal{H}^\beta_\infty(r_k^{-1}E \cap B_{1/2}) = r_k^{-\beta}\mathcal{H}^\beta_\infty(E \cap B_{r_k/2}) \geq 2^{-\beta} > 0. \tag{7.18}$$

Now suppose that $\mathcal{H}^\beta(A) = 0$. So for any $\varepsilon > 0$ let $\{B_j\}_{j \in \mathbb{N}}$ be a collection of balls such that $\mathcal{A} \subset \cup B_j$ and $\sum_{j \in \mathbb{N}} \operatorname{diam}(B_j)^\beta \leq \varepsilon$. Moreover, we have that for k large enough that

$$r_k^{-1}E \cap \overline{B_{1/2}} \subset \bigcup_{j \in \mathbb{N}} B_j.$$

Indeed if this were not the case then there would be a sequence in $\left(r_k^{-1}E \cap \overline{B}_{1/2}\right) \setminus \bigcup_{j \in \mathbb{N}} B_j$ whose limit would be in both $\mathcal{A}$ and $\overline{B}_{1/2} \setminus \bigcup_{j\in\mathbb{N}} B_j$. It then follows that

$$\mathcal{H}^\beta_\infty(r_k^{-1}E \cap \overline{B_{1/2}}) \leq \varepsilon,$$

contradicting (7.18) for small enough ε. $\square$

This proposition is the key technical tool we will use to establish the estimates on the size of the sets Σ^a_m for $m = 2, \ldots, n-2$.

7.6.2 Estimating the size of Σ_m^a

We will need to first study what happens when singular points accumulate which is the content of the following lemma.

Lemma 7.25. *Let $n \geq 3$ and $0 \in \Sigma_m^a$ where $m = \dim(L) \leq n-2$. Suppose there exists a sequence of singular points $x_k \to 0$ and radii $r_k \downarrow 0$ such that $|x_k| \leq \frac{r_k}{2}$ and*

$$\tilde{w}_{r_k} \rightharpoonup q \text{ in } W^{1,2}(B_1)$$

and $y_k = \frac{x_k}{2} \to y_\infty$. Then, $y_\infty \in L$ and $q(y_\infty) = 0$.

Proof. We first observe that $(u - f(0)p_0)(r_k y_k) = u(x_k) - r_k^2 f(0)p_0(y_k)$ and since $u(x_k) = 0$ and $(u - f(0)p_0)(rx) = o(r^2)$ we have that $p_0(y_\infty) = 0$. This shows that $y_\infty \in L$.

To show that $q(y_\infty) = 0$ we will apply Monneau monotonicity at the singular point x_k and with $p = p_0$, $r = r_k\rho$ for $\rho \in (0, 1/2)$. Namely, we will have that for all $\rho \in (0, 1/2)$

$$\begin{aligned}\rho^{-4} \int_{\partial B_1} &|u(x_k + r_k\rho\cdot) - f(0)p_0(r_k\rho\cdot)|^2 + C(r_k\rho)^\alpha r_k^4 \\ &\leq 2^4 \int_{\partial B_1} \left| u(x_k + \frac{r_k}{2}\cdot) - f(0)p_0(\frac{r_k}{2}\cdot)\right|^2 + C(\frac{r_k}{2})^\alpha r_k^4. \qquad (7.19)\end{aligned}$$

Now we would like to pass the limit here but we need to first see if $u(x_k + r_k\cdot) - f(0)p_0(r_k\cdot)$ converges and to what. To this end we note that we can rewrite this expression as

$$u(x_k + r_k\cdot) - f(0)p_0(r_k\cdot) = \tilde{w}_{r_k}(y_k + \cdot) + h_{r_k}^{-1}(f(0)p_0(x_k + r_k\cdot) - f(x_k)p_0(r_k\cdot))$$

where $h_{r_k} = \|u - f(0)p_0\|_{L^2(\partial B_1)}$. Observe that

$$h_{r_k}^{-1}(f(0)p_0(x_k + r_k\cdot) - f(x_k)p_0(r_k\cdot)) = b_k \cdot x + c_k + (f(0) - f(x_k))p_0(r_k x)h_{r_k}^{-1}$$

for some $c_k \in \mathbb{R}$ and $b_k \in \mathbb{R}^n$ and $b_k \perp L$. In the limit the term $(f(0)-f(x_k))p_0(r_k x)h_{r_k}^{-1}$ will go to zero since

$$(f(0) - f(x_k))p_0(r_k x)h_{r_k}^{-1} \leq C\frac{r_k^{2+\alpha}}{h_{r_k}},$$

and so we will focus on the convergence of the affine function $b_k \cdot x + c_k$.

Claim: There exists some constant $C \in \mathbb{R}$ so that $|b_k| + |c_k| \leq C$ for all $k \in \mathbb{N}$.

We will first show how to conclude using this claim. First there exists up to a subsequence limits b_∞ and c_∞ so that $b_k \cdot x + c_k \to b_\infty \cdot x + c_\infty$ uniformly. So passing the limit as $k \to \infty$ in (7.19) we obtain that (note that I have first rescaled the integrals and then divided the expression by h_{r_k} and so we again implicitly use $\frac{r^{2\gamma}}{H(r,w)} \downarrow 0$)

$$\rho^{-4} \fint_{\partial B_\rho} |q(y_\infty + x) + b_\infty \cdot x + c_\infty|^2 \leq 2^4 \fint_{\partial B_{1/2}} |q(y_\infty + x) + b_\infty \cdot x + c_\infty|^2.$$

As a consequence we have that, using the equivalence of norms in finite dimensional vector space of quadratic polynomials,

$$\|q(y_\infty) + b_\infty \cdot x + c_\infty\|_{L^\infty(B_\rho)} \le C\rho^2.$$

This says that $q(y_\infty) + b_\infty \cdot x + c_\infty$ behaves (at least) quadratically, and so in the directions of L, we must have that $\nabla_L q(y_\infty) = 0$, since $b_\infty \perp L$. Since q is homogeneous and $y_\infty \in L$ we conclude that $q(y_\infty) = 0$.

We now prove the claim.

Proof of Claim: We will use the truncated Algmren frequency applied at x_k with $p = p_0$. Note that $\phi^\gamma(\frac{r_k}{2}, u(x_k + \cdot) - f(x_k)p_0) \ge 2 - Cr_k^\varepsilon$. This means that

$$\frac{\left(\frac{r_k}{2}\right)^{2-n} \int_{B_{r_k/2}} |\nabla(u(x_k + \cdot) - f(x_k)p_0)|^2 + \gamma r_k^{2\gamma}}{\left(\frac{r_k}{2}\right)^{1-n} \int_{\partial B_{r_k/2}} |u(x_k + \cdot) - f(x_k)p_0|^2 + r_k^{2\gamma}} \ge 2 - Cr_k^\varepsilon.$$

We first divide the top and bottom of the left hand side by $\left(\frac{r_k}{2}\right)^{1-n} \int_{\partial B_{r_k/2}} |u(x_k + \cdot) - f(x_k)p_0|^2$ and obtain

$$\frac{\frac{\left(\frac{r_k}{2}\right)^{2-n} \int_{B_{r_k/2}} |\nabla(u(x_k+\cdot) - f(x_k)p_0)|^2}{\left(\frac{r_k}{2}\right)^{1-n} \int_{\partial B_{r_k/2}} |u(x_k+\cdot) - f(x_k)p_0|^2} + \frac{\gamma r_k^{2\gamma}}{H(r_k/2, u(x_k+\cdot) - f(x_k)p_0)}}{1 + \frac{r_k^{2\gamma}}{H(r_k/2, u(x_k+\cdot) - f(x_k)p_0)}} \ge 2 - Cr_k^\varepsilon.$$

Since $x_k \to 0$ we have again by our assumptions that $\frac{r_k^{2\gamma}}{H(r_k/2, u(x_k+\cdot) - f(x_k)p_0)} \downarrow 0$ as $k \to \infty$ and so

$$\lim_{k\to\infty} \frac{\left(\frac{r_k}{2}\right)^{2-n} \int_{B_{r_k/2}} |\nabla(u(x_k + \cdot) - f(x_k)p_0)|^2}{\left(\frac{r_k}{2}\right)^{1-n} \int_{\partial B_{r_k/2}} |u(x_k + \cdot) - f(x_k)p_0|^2} \ge 2.$$

Rescaling the integrals and using the definition of $\tilde{w}_{r_k}$ as before we have that

$$\frac{\left(\frac{r_k}{2}\right)^{2-n} \int_{B_{r_k/2}} |\nabla(u(x_k + \cdot) - f(x_k)p_0)|^2}{\left(\frac{r_k}{2}\right)^{1-n} \int_{\partial B_{r_k/2}} |u(x_k + \cdot) - f(x_k)p_0|^2} = \frac{1}{2} \frac{\int_{B_{1/2}} |\nabla(\tilde{w}_{r_k}(y_k + \cdot) + b_k \cdot x + c_k)|^2}{\int_{\partial B_{1/2}} |\tilde{w}_{r_k}(y_k + \cdot) + b_k \cdot x + c_k|^2}.$$

Suppose now that there did not exists such a C, then dividing the top and bottom of this fraction by $(|c_k| + |b_k|)^2$ we have that $\bar{c}_k = \frac{c_k}{(|c_k|+|b_k|)^2}$ and $\bar{b}_k = \frac{b_k}{(|c_k|+|b_k|)^2}$ are both bounded above by 1 and so converge (up to subsequences) to $\bar{c}_\infty$ and $\bar{b}_\infty$. Moreover, $\int_{B_{1/2}} |\nabla \tilde{w}_{r_k}(y_k + \cdot)|^2$ and $\int_{\partial B_{1/2}} |\tilde{w}_{r_k}(y_k + \cdot)|^2$ are controlled by $\|w_{r_k}\|^2_{W^{1,2}(B_1)}$ which is uniformly bounded and so $\frac{\tilde{w}_{r_k}(y_k+\cdot)}{(|c_k|+|b_k|)^2} \to 0$ and $\frac{\nabla \tilde{w}_{r_k}(y_k+\cdot)}{(|c_k|+|b_k|)^2} \to 0$. Putting all this together we obtain in the limit that

$$2 \leq \frac{1}{2} \frac{\int_{B_{1/2}} \left|\nabla(\bar{c}_\infty + \bar{b}_\infty \cdot x)\right|^2}{\int_{\partial B_{1/2}} \left|\bar{c}_\infty + \bar{b}_\infty \cdot x\right|^2} \leq 1,$$

a contradiction which proves the claim. □

□

We can now estimate the size of the sets Σ_m^a.

Proposition 7.26. *Let* $\Sigma_m^a = \{x_0 \in \Sigma_m : \ \phi(0^+, u - f(x_0)p_{x_0}) = 2\}$. *Then,*

a) if $n \geq 3$ *we have that* Σ_1^a *is a discrete set;*

b) if $n \geq 4$ *we have that* $\dim_{\mathcal{H}}(\Sigma_m^a) \leq m - 1$ *for* $2 \leq m \leq n - 2$

Proof. We first prove (a). With no loss of generality we will assume $0 \in \Sigma_1^a$ and we first note since $\phi^\gamma(0^+, u - f(0)p_0) = \phi(0^+, u - f(0)p_0) = 2 < \gamma$ we are in case b(ii) of Proposition 7.13. Now suppose there exists a sequence of singular points $x_k \to 0$ and define $r_k = 2|x_k|$. We then have up to subsequences that $\tilde{w}_{r_k} \to q$ in $L^2(B_1)$ and $y_k = \frac{x_k}{r_k} \to y_\infty \in L$ where $|y_\infty| = \frac{1}{2}$ and $q(y_\infty) = 0$. However, since $\lambda_* = 2$ we have that in an appropriate system of co-ordinates

$$p_0(x) = \frac{1}{2} \sum_{i=2}^{n} \mu_i x_i^2, \quad q(x) = \nu \sum_{i=2}^{n} x_i^2 - \nu_1 x_1^2,$$

where $\nu_1 = (n-1)\nu > 0$ and $\sum \mu_i = 1$. Now, $y_\infty \in L$ and $\dim(L) = 1$ while $q(y_\infty) = 0$. As a consequence of the homogeneity of q, we must have that $q|_L \equiv 0$ contradicting the fact that $\nu_1 > 0$. This proves (a).

We now prove (b) and so suppose that for some $\beta > m - 1$, $0 < \mathcal{H}_\infty^\beta(\Sigma_m^a) < +\infty$ and let x_0 be a density point, that is, there exists some sequence $r_k \downarrow 0$ such that

$$r_k^{-\beta} \mathcal{H}_\infty^\beta(\Sigma_m^a \cap B_{r_k}(x_0)) \geq 1.$$

With no loss of generality we will assume that $x_0 = 0$. We note that by Proposition 7.13 that (up to extracting a subsequence) that $\tilde{w}_{r_k} \to q$ in $L^2(B_1)$ and since by assumption $\lambda_* = 2$ we have that in an appropriate system of co-ordinates

$$p_0(x) = \frac{1}{2} \sum_{i=m+1}^{n} \mu_i x_i^2, \quad q(x) = \nu \sum_{i=m=1}^{n} x_i^2 - \sum_{i=1}^{m} \nu_i x_i^2,$$

where $\sum \nu_i = (n-m)\nu > 0$ and $\sum \mu_i = 1$. Moreover by Proposition 7.24 we have that $\mathcal{H}_\infty^\beta(\mathcal{A}_{\Sigma_m^a, \{r_k\}}) > 0$.

We now observe that $\mathcal{A}_{\Sigma^a_m,\{r_k\}} \subset \overline{B_1} \cap L \cap \{q = 0\}$ as a consequence of Lemma 7.25. Indeed, any accumulation point $z \in \mathcal{A}_{\Sigma^a_m,\{r_k\}}$ must satisfy $\frac{z}{2} \in L$ and $q(\frac{z}{2}) = 0$ and since both p_0 and q are homogeneous we have that $z \in L \cap \{q = 0\}$. Furthermore, $\dim(L \cap \{q = 0\}) \leq m - 1$ since otherwise we would have that $q \equiv 0$ on L contradicting that $\nu > 0$. Consequently, $\mathcal{H}^\beta_\infty(L \cap \{q = 0\} \cap \overline{B_1}) = 0$ since $\beta > m - 1$. However,

$$0 < \mathcal{H}^\beta_\infty(\mathcal{A}_{\Sigma^a_m,\{r_k\}}) \leq \mathcal{H}^\beta_\infty(L \cap \{q = 0\} \cap \overline{B_1}) = 0,$$

a contradiction and so $\dim_{\mathcal{H}}(\Sigma^a_m) \leq m - 1$. $\square$

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