The Many Triumphs of De-Giorgi's Improvement of Flatness

Anthony Salib

1 Introduction

This blog post will be the beginning of a new series - a short survey of how De-Giorgi's improvement of flatness technique has been used in different free boundary problems. To get there we will first need a conceptual understanding of the method which is the purpose of this first post. We follow very closely the treatment contained in [1].

This technique has it's beginnings in the regularity theory of minimal surfaces, which is the subject of today's post. Specifically, we will sketch the proof of the following (vaguely worded) statement:

If a minimal surface in $\mathbb{R}^3$ is close to a plane, then it is smooth.

Let's make this less vague.

For simplicity we will confine ourselves to surfaces in $\mathbb{R}^3$ that can be expressed as a graph $u(x, y)$ - we will not enter into the different ways in which one can define the notion of a "surface". However, we will mention that the theory of minimal graphs has it's own treatment, but we will not discuss this here (in particular minimal graphs are *always* smooth).

Now that we have solidified what surfaces we are considering we must define what a minimal surface is. Naturally, since u is a graph, one can compute it's area in any domain, say B_1, by integration. Now suppose I *deform* u in any compact set in B_1 then compute the area of this deformation. If u has an area that is smaller than any such deformation we say that u minimises the area functional. By standard arguments in the calculus of variations one can show that if u minimises the area functional then it must necessarily satisfy

$$\int_{B_1} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \cdot \nabla\varphi = 0 \quad \forall\varphi \in C_c^\infty(B_1). \tag{1.1}$$

Integrating by parts one can easily see that the corresponding PDE is

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)=0. \tag{1.2}$$

Equation (1.2) is known as the Minimal Surface Equation (MSE) in divergence form. I emphasise the 'divergence form' here since we can also treat the MSE in non-divergence form which is written as

$$M(u) := (1+|\nabla u|^2)\Delta u + \nabla u^T D^2 u \nabla u = 0. \tag{1.3}$$

If you're familiar with the two theories of second order Elliptic PDE in divergence form and non-divergence form, i.e. De-Giorgi-Nash and Krylov-Safanov Theorems, then it should come as no surprise that treating this problem using (1.2) and (1.3) use vastly different techniques. Equation (1.2) leads one to use 'energy methods' while Equation (1.3) lends one to 'viscosity methods'. Today's post is about the 'energy method' while the rest of this survey will concern the 'viscosity' approach.

Now we will stop here and try to understand the MSE. A good first point is to look for solutions and just by looking at the equation, we see that planes are solutions since they satisfy $D^2u = 0$. In particular, planes are examples of smooth solutions. Now let's try to understand what happens at the level of the MSE when we perturb these planes very slightly. Specifically, if u is a plane we will consider $u+\varepsilon w$ for any smooth w, and we ask what must w satisfy in order for the perturbation to still satisfy the MSE, up to first order terms in ε. That is we ask w to be a solution to the linearised equation $L_u(w) = 0$ where

$$L_u(w) := \lim_{\varepsilon\to 0}\left(\frac{M(u+\varepsilon w) - M(u)}{\varepsilon}\right).$$

Now since the MSE is a geometric equation about the curvature of a surface, it is invariant under rotations. So given any plane, we can always rotate our picture so the plane coincides with $u = 0$. Computing the linearised operator around $u = 0$ we find that

$$\Delta w = 0$$

and so w must be harmonic. We can now see why one would expect our vague statement to be true. Indeed, being close to a plane we could assume that our surface is a small perturbation like $u+\varepsilon w$, and if this is a minimal surface w must be harmonic. Since harmonic functions are smooth, $u + \varepsilon w$ is smooth.

This is far from a proof, but it highlights an important element of the method of improvement of flatness. If u is close to a plane, then you expect to be close - in some sense - to a harmonic function!

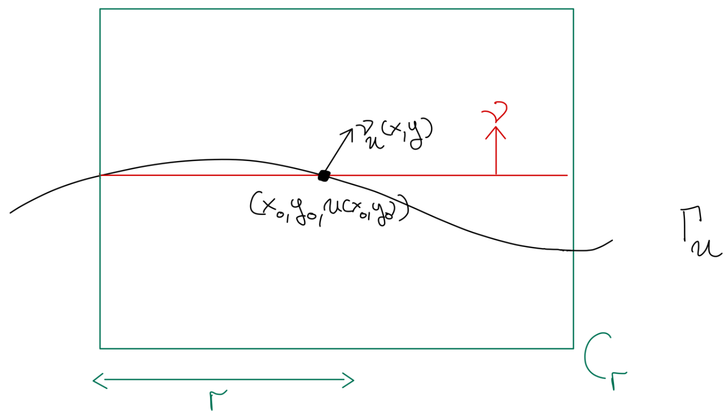

Figure 1: The graph of u is close to the horizontal plane in C_r

We will now try to put this idea in the context of the proof. The first thing we need to do is quantify what we mean by being close to a plane, and following the original proof of De-Giorgi we will do this via energy methods. Consider the picture in Figure 1, we have drawn the horizontal plane going through a point $(x_0, y_0, u(x_0, y_0))$ in the graph of u, Γ_u. We have also drawn a cyllinder around (x_0, y_0) which we labelled C_r in which we will analyse our surface. The subscript r is to label the width of the cylinder and will be important quantity to keep track of.

Now the horizontal plane has normal vector ν while the graph has normal vector $\nu_u(x, y)$ and so one way to measure how different our surface is to the horizontal plane in C_r is to measure the difference between ν_u and ν in C_r. This leads us to define the *excess* in the cylinder C_r as

$$e((x_0, y_0), C_r, \nu) = \int_{\Gamma_u \cap C_r} \frac{|\nu_u(x, y) - \nu|^2}{2}.$$

From Figure 1 we can notice around (x_0, y_0) that there is a different plane that will decrease the excess in a smaller cylinder. If one would describe this process they would say by *tilting* the plane slightly we could *improve the excess* in a smaller cylinder around (x_0, y_0), see Figure 2. Of course decreasing the cylinder would decrease the excess even if we kept the original plane! So the decrease has to be in proportion with the decrease in the cylinder size.

This is the spirit of the improvement of flatness methods and it's applications. Although it is clear from the very nice pictures I drew that tilting can improve the excess, proving this requires some work and will tie us back to our discussion of harmonic functions. At this point we precisely formulate the improvement of excess by tilting Theorem.

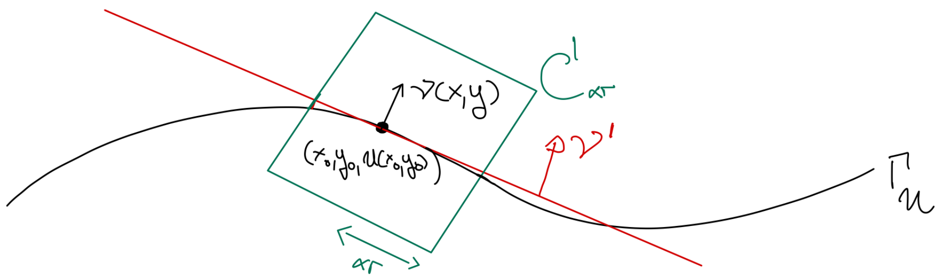

Figure 2: The graph of u is closer to the tilted horizontal plane in $C'_{\alpha r}$

Theorem 1.1. *Suppose* Γ_u *is a minimal surface in some cylinder* C_r *centred at a point* $(x_0, y_0, u(x_0, y_0))$ *and satisfies*

$$e((x_0, y_0), C_r, \nu) \leq \varepsilon$$

for some $\nu \in \mathbb{S}^2$ *and* $\varepsilon \leq \varepsilon_1$ *where* ε_1 *is a universal constant. There exists a* $\nu' \in \mathbb{S}^2$ *such that*

$$e((x_0, y_0), C_{\alpha r}, \nu) \leq C(n)\alpha^2 \varepsilon,$$

for some α *small and universal.*

Our goal of showing that u is smooth follows from this Theorem. Suppose that at smaller and smaller scales (that is looking at smaller cylinders around $(x_0, y_0, u(x_0, y_0))$ we can always find a normal vector ν_k so that the difference between ν_u and ν_k decreases *geometrically* in k. We will then have a Cauchy sequence of vectors $\{\nu_k\}_{k\in\mathbb{N}}$ that will converge to the normal vector of u at $(x_0, y_0, u(x_0, y_0))$. Hence u will be differentiable at (x_0, y_0) - actually u will be $C^{1,\alpha}$ but I will show this another time. Once you have that u is $C^{1,\alpha}$, Schauder theory applied to the MSE and a standard bootstrap argument yields the smoothness. Again I will discuss all this in the next blog post, for now, we wish to understand the main ideas of the proof of Theorem 1.1.

For simplicity we will take the plane to be the horizontal plane, that is, $\nu = e_3$. A major step is to now prove that u is Lipschitz, and if the excess is small enough, it satisfies $|\nabla u| \leq 1$. Once this is proved, writing the outward unit normal to Γ_u as

$$\nu_u = \frac{(-\nabla u, 1)}{\sqrt{1 + |\nabla u|^2}}.$$

we can compute that

$$\begin{aligned}\frac{|\nu_u - e_3|^2}{2} &= 1 - (\nu_u \cdot e_3)\\ &\geq \frac{1 - (\nu_u \cdot e_n)^2}{2}\\ &= \frac{1}{2}\frac{|\nabla u|^2}{1 + |\nabla u|^2}.\end{aligned}$$

Combing this with the fact that $|\nabla u| \leq 1$ we get that

$$\frac{1}{2}|\nabla u|^2 \leq 2\frac{|\nu_u - e_3|^2}{2}$$

and so

$$\frac{1}{2}\int_{\pi(\Gamma_u \cap C_r)} |\nabla u|^2 \leq 2e((x_0, y_0), C_r, e_3) \tag{1.4}$$

where $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ is the projection down onto the $x - y$ plane. However we began by assuming that the excess is small and so what we have found is that

$$\frac{1}{2}\int_{\pi(\Gamma_u \cap C_r)} |\nabla u|^2 \leq 2\varepsilon.$$

Now using Equation (1.1) we see that for any $\varphi \in C_c^\infty(\pi(\Gamma_u \cap C_r))$ that

$$\left|\int_{\pi(\Gamma_u \cap C_r)} \nabla u \cdot \nabla\varphi\right| = \left|\int_{\pi(\Gamma_u \cap C_r)} \nabla u \cdot \nabla\varphi - \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \cdot \nabla\varphi\right|$$

and so using $\sqrt{1 + |\nabla u|^2} \leq 1 + \frac{|\nabla u|^2}{2}$ and the fact that $|\nabla u| \leq 1$ again we find

$$\left|\int_{\pi(\Gamma_u \cap C_r)} \nabla u \cdot \nabla\varphi\right| \leq 2\|\nabla\varphi\|_{L^\infty}\varepsilon \quad \forall\varphi \in C_c^\infty(\pi(\Gamma_u \cap C_r)).$$

This is saying that if u has small excess then it is *almost harmonic*. Finally, being almost harmonic in the above sense means that you are close in L^2-norm to a harmonic function v. This is a key part of the proof and we formulate it in the following Lemma.

Lemma 1.2. *For every $\tau > 0$ there exists an $\sigma > 0$ so that if u satisfies for all $\varphi \in C_c^\infty(B_1)$*

$$\int_{B_1} |\nabla u|^2 \leq 1, \quad \textit{and} \quad \left|\int_{B_1} \nabla u \cdot \nabla\varphi\right| \leq \|\nabla\varphi\|_{L^\infty}\sigma,$$

then there exists a harmonic function v such that

$$\int_{B_1} |\nabla v|^2 \leq 1, \quad \textit{and} \quad \int_{B_1} |v - u|^2 \leq \tau.$$

Proof. The proof of this fact is by compactness and goes as follows. Suppose there exists a $\tau > 0$ and a sequence of functions $\{u_k\}_{k\in\mathbb{N}}$ satisfying

$$\int_{B_1} |\nabla u|^2 \leq 1, \quad \text{and} \quad \left|\int_{B_1} \nabla u \cdot \nabla \varphi\right| \leq \|\nabla \varphi\|_{L^\infty} \frac{1}{k},$$

but for every harmonic function v satisfying

$$\int_{B_1} |\nabla v|^2 \leq 1$$

we have

$$\int_{B_1} |u_k - v|^2 \geq \tau.$$

Now define the functions $w_k = u_k - \fint_{B_1} u_k$ and note that by the Poincaré inequality and reflexivity of L^2 we have that up to a subsequence $w_k \to w$ and $\nabla w_k \rightharpoonup \nabla w$ in L^2 with $\|\nabla w\|_{L^2} \leq 1$. Moreover since

$$\left|\int_{B_1} \nabla w \cdot \nabla \varphi\right| \leq \left|\int_{B_1} (\nabla w - \nabla w_k) \cdot \nabla \varphi\right| + \frac{\|\nabla \varphi\|_{L^\infty}}{k}$$

letting $k \to \infty$ we see that w is harmonic. Now since $w + \fint_{B_1} u_k$ is harmonic with $\|\nabla w\|_{L^2} \leq 1$ we have by assumption for all $k \in \mathbb{N}$ that

$$\tau \leq \int_{B_1} \left|u_k - (w + \fint_{B_1} u_k)\right|^2 = \int_{B_1} |w_k - w|^2$$

which is a contradiction since the right hand side goes to zero as $k \to \infty$. □

So now we have found the harmonic function that is close to u! The final thing to do is use derivative estimates for harmonic functions and a Taylor expansion to show that v is close in L^2 to its linear approximation at (x_0, y_0) in a smaller cylinder, $C_{\alpha r}$, around $(x_0, y_0, u(x_0, y_0))$. Taking the linear approximation of v as your new plane will then decrease your excess. This final step is quite involved in this energy method setting, but will see the details in next weeks blog in the different setting of 'viscosity methods'.

Although our simplifying assumption that u is a graph hid many of the difficulties involved in the argument, the spirit of the proof remains the same. The key take away is that minimality and being close to a plane really forces you to be close to a harmonic function. Since harmonic functions have derivatives of all orders controlled, this harmonic function will be very close to its linear approximation in a small domain, and so u must be very close to it as well.

Next week we will discuss the viscosity method approach that was pioneered by Savin in his celebrated paper [2].

References

[1] Francesco Maggi, *Sets of finite perimeter and geometric variational problems: an introduction to geometric measure theory*, Cambridge University Press, 2012. ↑1

[2] Ovidiu Savin, *Phase transitions, minimal surfaces, and a conjecture of De Giorgi*, Current developments in mathematics **2009** (2009), no. 1, 59–114. ↑6