

2 Savin's Approach

Today we begin a more precise discussion following last weeks heuristic. We follow the seminal work of Savin in [2] and we prove

Theorem 2.1 (De Giorgi). *There exists a universal ε_0 so that if ∂E is a minimal surface in B_1 , $0 \in \partial E$ and*

$$\partial E \cap B_1 \subset \{|x_n| \leq \varepsilon\}$$

for $\varepsilon \leq \varepsilon_0$ then ∂E is an analytic surface inside $B_{1/2}$.

As discussed, the proof of this Theorem follows from

Theorem 2.2 (Improvement of Flatness). *Suppose ∂E is a minimal surface in B_1 , $0 \in \partial E$ and*

$$\partial E \cap B_1 \subset \{|x_n| \leq \varepsilon\}.$$

There exists an $\varepsilon_0(n)$ universal so that if $\varepsilon \leq \varepsilon_0$ there exists $\nu \in \mathbb{S}^{n-1}$ such that

$$\partial E \cap B_{r_0} \subset \left\{ |x \cdot \nu| \leq \frac{\varepsilon}{2} r_0 \right\}$$

for some universal constant r_0 .

Throughout the proof we will denote points in \mathbb{R}^{n-1} as x' so that $x = (x', x_n) \in \mathbb{R}^n$ and we denote balls in \mathbb{R}^{n-1} as B' . This allows us to view ∂E as a multivalued graph over B'_1 . The proof is based on a viscosity approach and so we first observe that minimal surfaces satisfy the minimal surface equation in non-divergence form in the viscosity sense. That is, if P is a paraboloid that touches ∂E from below at x'_0 then we have

$$\Delta P(x'_0) + (\nabla P(x'_0))^T D^2 P(x'_0) \nabla P(x'_0) \leq 0$$

and the opposite inequality if it touches it from above.

In order to prove Theorem 1.1 we need the following Harnack inequality for flat minimal surfaces. The proof of this Theorem is quite involved and is very much in the same spirit as that of Krylov-Safanov Harnack inequality - we refer to [2, Section 6] for the proof.

Theorem 2.3. *Suppose E is a minimal surface in B_1 and that*

$$\partial E \cap B_1 \subset \{|x_n| \leq \varepsilon\}.$$

There exists an $\varepsilon_1(n)$ so that if $\varepsilon \leq \varepsilon_1(n)$ then

$$\partial E \cap B_{1/2} \subset \{|x_n| \leq \varepsilon(1 - \eta)\}$$

where $\eta > 0$ is a small universal constant.

We now show how this Harnack inequality implies the improvement of flatness theorem. The remarkable aspect of this proof is that it is in the same spirit of the proof of Hölder continuity of harmonic functions, that is

$$\text{Harnack Inequality} \implies \text{Oscillation decay} \implies \text{Hölder continuity}.$$

The key difference is that the final step is not Hölder inequality, rather we are led to a Hölder continuous function that is harmonic! We now give the details.

Proof of 1.1. We argue by compactness and suppose that no such ε_0 exists. Now take a sequence of minimal surfaces ∂E_k containing the origin such that

$$\partial E_k \cap B_1 \subset \{|x_n| \leq \varepsilon_k\}$$

where $\varepsilon_k \rightarrow 0$ as $k \rightarrow \infty$. For every $x_0 \in B_{1/2}$ we have that $|(x - x_0) \cdot e_n| \leq 2\varepsilon_k$ for all $x \in \partial E_k \cap B_{1/2}(x_0)$ and so if $4\varepsilon_k \leq \varepsilon_1(n)$, we can apply Theorem 2.3 in $B_{1/2}(x_0)$ to obtain

$$\partial E_k \cap B_{1/4}(x_0) \subset \{|(x - x_0) \cdot e_n| \leq 2\varepsilon_k(1 - \eta)\}.$$

We keep iterating Theorem 2.3 to obtain that

$$\partial E_k \cap B_{2^{-m}}(x_0) \subset \{|(x - x_0) \cdot e_n| \leq 2\varepsilon_k(1 - \eta)^{m-1}\} \quad (2.1)$$

which is possible as long as

$$2^m \varepsilon_k (1 - \eta)^{m-2} \leq \varepsilon_1(n). \quad (2.2)$$

We now define the sets

$$A_k = \left\{ \left(x', \frac{x_n}{\varepsilon_k} \right) : x \in \partial E_k \cap B_{1/2} \right\},$$

and from (2.1) we have the oscillation decay of the multivalued graphs A_k . Specifically we have for all $x_0 \in B_{1/2}$

$$A_k \cap \{|x' - x'_0| \leq 2^{-1-m}\} \subset \{|(x - x_0) \cdot e_n| \leq 2(1 - \eta)^{m-1}\}. \quad (2.3)$$

From (2.3) we can conclude that

$$|A_k(x') - A_k(y')| \leq C \left(|x' - y'| + C \varepsilon_k^\gamma \right)^\beta$$

where $\beta = -\log_2(1 - \eta)$ and $\gamma = (1 - \beta)^{-1}$. By the Arzela-Ascoli Theorem, A_k converge up to a subsequence to the graph of a Hölder continuous function $w(x')$ in $B'_{1/2}$.

We show that w is harmonic in $B'_{1/2}$ in the viscosity sense. Let $P(x')$ be a quadratic polynomial that touches w from below at some point $x'_0 \in B'_{1/2}$. By the uniform

convergence of the A_k to w we have that for some k large, $\varepsilon_k P(x') + c$ touches ∂E_k from below. To see this one needs to find the point x'_1 satisfying

$$\max_{\overline{B'}_{1/2}} \{ \varepsilon P(x') - \delta(k) |x' - x'_0|^2 - \partial E_k(x') \} = \varepsilon P(x'_1) - \delta(k) |x'_1 - x'_0|^2 - \partial E_k(x'_1).$$

where $\delta(k) > 0$ is a constant depending on k and we are treating ∂E_k as a multivalued graph over \mathbb{R}^{n-1} . For $\delta(k)$ large enough $x'_1 \notin \partial B'_{1/2}$ since by uniform convergence of the A_k to w there exists a $\gamma(k)$ so that $|\partial E_k - w| \leq \gamma(k)$. If then $x'_1 \in \partial B'_{1/2}$, and letting $r = \text{dist}(x'_0, \partial B'_{1/2})$, we would have

$$\gamma(k) - \delta(k)r^2 \geq \varepsilon P(x'_0) - \partial E_k(x'_0) \geq -\gamma(k)$$

which is a contradiction for $\delta(k) \geq 4 \frac{\gamma(k)}{r^2}$. Since ∂E_k is minimal we have that

$$\varepsilon_k \Delta P + \varepsilon_k^3 (\nabla P)^T D^2 P \nabla P \leq 0,$$

and so dividing by ε_k and sending $k \rightarrow \infty$ we obtain that $\Delta P \leq 0$. This shows that $\Delta w \leq 0$ in the viscosity sense. A similar argument with a paraboloid touching w from above yields that $\Delta w \geq 0$ in the viscosity sense and so, w is harmonic in $B'_{1/2}$. Since w is harmonic, $w(0) = 0$ and $|w| \leq 1$ a Taylor expansion yields

$$|w(x') - \nabla w(0) \cdot x'| \leq 4C(n)r_o^2$$

for $|x'| \leq 2r_o$. Choosing r_o small enough we have that

$$|w(x') - \nabla w(0) \cdot x'| \leq \frac{r_o}{4}$$

and by uniform convergence, we have for k large enough so that $|A_k(x') - w(x')| \leq \frac{r_o}{4}$ for $|x'| \leq r_o$, that

$$A_k \cap \{|x'| \leq r_o\} \subset \left\{ |x_n - x' \cdot \nabla w(0)| \leq \frac{r_o}{2} \right\}.$$

Then setting $\nu = (-\varepsilon_k \nabla w(0), 1) |(-\varepsilon_k \nabla w(0), 1)|^{-1}$ we have that

$$\partial E_k \cap B_{r_o} \subset \left\{ |x \cdot \nu| \leq \varepsilon_k \frac{r_o}{2} \right\}.$$

This is a contradiction and hence we have proved the Theorem. \square

We now give the

Proof of Theorem 2.1. We will show that ∂E is $C^{1,\sigma}$ around every point $x_0 \in \partial E \cap B_{1/2}$ and the analyticity will follow from Schauder estimates for the MSE. With no loss

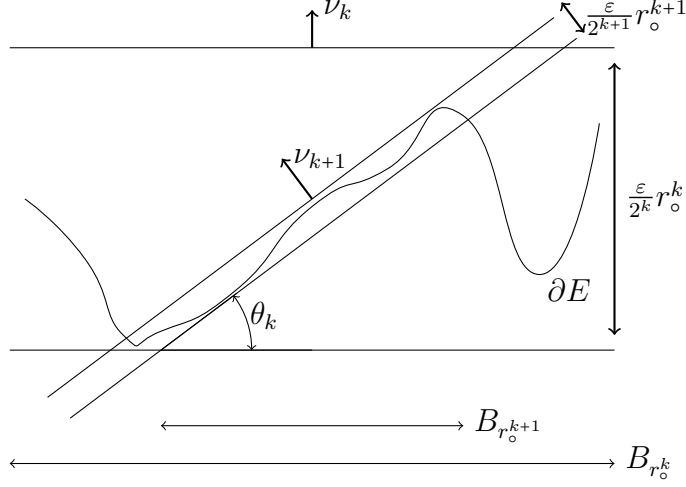


Figure 3: The largest possible θ_k

of generality assume $x_0 = 0$ and iteratively apply Theorem 1.1 around 0 to obtain a sequence of normal vectors $\{\nu_k\}_{k \in \mathbb{N}}$ satisfying

$$\partial E \cap B_{r_o^k} \subset \left\{ |x \cdot \nu_k| \leq \frac{\varepsilon}{2^k} r_o^k \right\}.$$

We now estimate the angle, θ_k , between two consecutive vectors ν_k and ν_{k+1} . From Figure 3 we obviously must have that

$$|\tan(\theta_k)| \leq \frac{1}{r_o} \frac{\varepsilon}{2^k}$$

and so

$$|\theta_k| \leq C \frac{\varepsilon}{2^k},$$

where $C = C(r_o)$ is a universal constant since r_o is universal. Using the identity $\cos(\theta) \geq 1 - \theta^2$ we then obtain that

$$|\nu_{k+1} - \nu_k| \leq C \frac{\varepsilon}{2^k}.$$

We now observe that this implies that

$$\lim_{k \rightarrow \infty} \nu_k = \nu_1 + \sum_{k \in \mathbb{N}} \nu_{k+1} - \nu_k$$

exists and is well defined. Calling this limit $\nu(0)$ we have that

$$|\nu(0) - \nu_k| \leq C \frac{\varepsilon}{2^k}$$

and so for all $x \in \partial E \cap B_{r_\circ^k}$ we have

$$|x \cdot \nu(0)| \leq |x \cdot (\nu(0) - \nu_k)| + |x \cdot \nu_k| \leq C\varepsilon r_\circ^{k(1+\sigma)}$$

where $\frac{1}{2} = r_\circ^\sigma$. So we obtain that

$$\partial E \cap B_{r_\circ^k} \subset \left\{ |x \cdot \nu(0)| \leq C\varepsilon r_\circ^{k(1+\sigma)} \right\}$$

and by letting $k \rightarrow \infty$ we immediately see that ∂E only attains one value over $0 \in \mathbb{R}^{n-1}$. Doing this over every point in $B'_{1/2}$ we have that ∂E is a graph over $B'_{1/2}$. We call this graph u and conclude that for every $x_0 \in \partial E \cap B_{1/2}$ we have

$$\|u - u(x'_0) - (x - x_0) \cdot \nu^\perp(x_0)\|_{L^\infty(B_{r_\circ^k}(x_0))} \leq C\varepsilon r_\circ^{k(1+\sigma)} \quad (2.4)$$

for all $k \geq 1$. Dividing Equation (2.4) by r_\circ^k and letting $k \rightarrow \infty$ we automatically see that $\nabla u(x_0) = \nu^\perp(x_0)$ and so we obtain again from (2.4) that

$$\|u(x') - u(x'_0) - \nabla u(x'_0) \cdot (x' - x'_0)\|_{L^\infty(B'_{r_\circ^k}(x'_0))} \leq C\varepsilon r_\circ^{k(1+\sigma)}. \quad (2.5)$$

Now fix any two points $x'_1, x'_2 \in B'_{1/2}(0)$ so that $|x'_1 - x'_2| \leq r_\circ$. Moreover fix some $k_\circ \in \mathbb{N}$ large enough so that

$$r_\circ^{k_\circ+1} \leq |x'_1 - x'_2| \leq \frac{1}{8} r_\circ^{k_\circ}.$$

Now let

$$x'_3 = x'_2 + \frac{r_\circ^{k_\circ}}{8} \frac{\nabla u(x'_1) - \nabla u(x'_2)}{|\nabla u(x'_1) - \nabla u(x'_2)|}$$

so that $|x'_1 - x'_3| \leq \frac{1}{4} r_\circ^{k_\circ}$ and $|x'_2 - x'_3| \leq \frac{1}{8} r_\circ^{k_\circ}$. By (2.5) we then have that

$$|(\nabla u(x'_1) - \nabla u(x'_2)) \cdot (x'_3 - x'_2)| \leq C\varepsilon r_\circ^{\sigma(k_\circ+1)}$$

and since $x'_3 - x'_2$ is parallel to $\nabla u(x'_1) - \nabla u(x'_2)$ we have

$$\frac{|\nabla u(x'_1) - \nabla u(x'_2)|}{r_\circ^{k_\circ\sigma}} \leq C\varepsilon.$$

However, $\frac{|x'_1 - x'_0|^\sigma}{r_\circ^\sigma} \geq r_\circ^{k_\circ\sigma}$ and so we conclude

$$\frac{|\nabla u(x'_1) - \nabla u(x'_2)|}{|x'_1 - x'_2|^\sigma} \leq \frac{1}{r_\circ^\sigma} C\varepsilon < C.$$

If $|x'_1 - x'_2| > r_\circ$ then one can apply the triangle inequality for a sequence of points separated less than r_\circ apart connecting x'_1 and x'_2 , and consequently, we obtain that ∂E is a $C^{1,\sigma}$ graph in $B_{1/2}$. \square

References

- [1] Francesco Maggi, *Sets of finite perimeter and geometric variational problems: an introduction to geometric measure theory*, Cambridge University Press, 2012. [↑1](#)
- [2] Ovidiu Savin, *Phase transitions, minimal surfaces, and a conjecture of De Giorgi*, Current developments in mathematics **2009** (2009), no. 1, 59–114. [↑7](#)