Nash-Moser-De Giorgi

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In this short note we are concerned with proving

Theorem 1.1. *Let* $\Omega \subset\subset \mathbb{R}^n$ *and* $u \in H^1(\Omega)$ *be a weak solution of*

$$\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) = 0, \tag{1.1}$$

where a_{ij} *are bounded, measurable and uniformly elliptic. Then* $u \in C^{\alpha}_{loc}(\Omega)$ *for some* $\alpha > 0$.

The method we adopt here is based on the methods of Campanato. We will show that our weak solution u lies in the Campanato space $\mathcal{L}^{2,\lambda}(\Omega)$ for some $\lambda > 0$, that is

$$\mathcal{L}^{2,\lambda}(\Omega) = \left\{u \in L^2(\Omega) : \sup_{0<r<\min\{1,dist(x_0,\partial\Omega)\},x_0\in\Omega} r^{-\lambda}\int_{\Omega\cap B_r(x_0)} |u - u_{x_0,r}|^2 < \infty\right\},$$

where

$$u_{x_0,r} = \frac{1}{|\Omega \cap B_r(x_0)|}\int_{\Omega\cap B_r(x_0)} u.$$

Then thanks to the following theorem, this will imply that our solution is Hölder continuous.

Theorem 1.2 (Campanato)**.** *If* $\lambda > n$ *then*

$$\mathcal{L}^{2,\lambda} \hookrightarrow C^{0,\alpha}$$

where $\alpha = \frac{\lambda-n}{2}$.

Remark 1.3. *So that these averages that we are taking don't blow up and things remain meaningful, it is necessary to assume that Ω is of type A, that is there exists some $A > 0$ such that for all $r > 0$ and $x_0 \in \Omega$ there holds $|\Omega \cap B_r(x_0)| \geq Ar^n$. This is not too important in what follows.*

We know (by the Cacciopolli inequality) that for u as in theorem 1.1 there exists some $\theta \in (0,1)$ such that for any $0 < R \leq 1$ there holds

$$\int_{B_{R/2}(x_0)} |\nabla u|^2 \leq \theta \int_{B_R(x_0)} |\nabla u|^2, \tag{1.2}$$

and coupling this with Campanato's embedding theorem is enough to give the proof of theorem 1.1 in $n = 2$.

Proof of theorem 1.1 when $n = 2$. For any $x_0 \in \Omega$ and $R > 0$ such that $0 < r < R < \min\{1, dist(x_0, \partial\Omega)\}$ we define the function

$$\phi(r) = \int_{B_r(x_0)} |\nabla u|^2.$$

We now note that by the Poincaré inequality have that

$$\int_{B_r(x_0)} |u - u_{x_0,r}|^2 \leq Cr^2 \int_{B_r(x_0)} |\nabla u|^2 = Cr^2\phi(r).$$

It is then enough to control $\phi(r)$ by r^α for some α small. However, this is a consequence of (1.2). Explicitly, we take $\alpha > 0$ such that $\theta = 2^{-\alpha}$ and for any $r > 0$ choose $k \in \mathbb{N}$ such that

$$2^{-k-1} < \frac{r}{R} < 2^{-k},$$

then by monotonicity of ϕ we have that

$$\begin{aligned}
\phi(r) &\leq \phi(2^{-k}R) \\
&\leq \theta^k \phi(R) \\
&\leq 2^{-k\alpha}\phi(R) \\
&\leq (\frac{2r}{R})^\alpha \phi(R) \\
&\leq Cr^\alpha,
\end{aligned}$$

where $C = (\frac{2}{R})^\alpha \phi(R)$. Hence we arrive at

$$\int_{B_r(x_0)} |u - u_{x_0,r}|^2 \leq C r^{2+\alpha},$$

which shows that $u \in \mathcal{L}^{2,2+\alpha}(B_r(x_0)) \hookrightarrow C^{0,\alpha/2}(B_r(x_0))$.

□

It is worth at this point taking a step back and looking at what happened. The quantity $\phi(r)$ is dimensionless for $n = 2$, this meaning that it remains invariant under rescaling. Now since any dimensionless quantity scales the same as the oscillation, (since the oscillation is invariant under rescaling), there is a chance that we can link the two concepts, our dimensionless quantity and the oscillation. Note that there is only a chance, not a guarantee that this works, it really all depends on the dimensionless quantity we look at. Now, the Caccioppoli inequality gave us that $\phi(r)$ decays geometrically, (1.2), and if we can somehow link this to the oscillation we would prove that the oscillation decays (and hence Hölder regularity). What our above proof has shown is that we can link this quantity $\phi(r)$ with the oscillation via the Campanato embedding theorem.

For $n \geq 3$ a variant of the above method can be used, however, we obviously need to find the right dimensionless quantity. For any $x_0 \in \mathbb{R}^n$ and $R > 0$ we introduce the fundamental solution of (1.1) in the ball $B_R(x_0)$, $g(x, x_0)$, that is g satisfies

$$\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) = \delta(x_0) \tag{1.3}$$

Using the ideas of Nash, Fabes and Strook achieved in [1] the a-priori bounds

$$\frac{1}{C(n)}|x - x_0|^{2-n} \leq g(x, x_0) \leq C(n)|x - x_0|^{2-n}. \tag{1.4}$$

(Note that you need to integrate their heat kernel bounds in time in order to achieve these bounds here).

This is then enough to conclude theorem 1.1 for $n \geq 3$. As was done in [1], (1.4) gives rise to a weak Harnack inequality which then gives the oscilation decay and finally Hölder regularity in the usual way. Indeed, the hard work was done in showing (1.4), however we outline another way to conclude the Hölder continuity using Campanato's methods. We first redefine

$$\phi(r) = \int_{B_r(x_0)} |\nabla u|^2 g(x, x_0), \tag{1.5}$$

and note that it is dimensionless in $\mathbb{R}^n$.

Then by (1.4) there holds $g(x, x_0) \geq Cr^{2-n}$ in $B_r(x_0)$. This coupled with the Poincaré inequality yields

$$\int_{B_r(x_0)} |u - u_{x_0,r}|^2 \leq Cr^m \phi(r).$$

Now proving that $\phi(r)$ is again a contraction requires testing (1.1) with the right testing function. We don't perform all the details here (see [2, Theorem 2.8] for the details, in fact, this is the inspiration for this treatment) but we describe the testing function. First define $\tilde{u} = \fint_{B_R \setminus B_{R/2}} u$ and then using a standard mollifying sequence $(\rho_\varepsilon)_{\varepsilon>0}$ regularise the fundamental solution

$$g_\varepsilon(x, x_0) = \int G(x, x_0 - z)\rho_\varepsilon(z)dz.$$

Now take a cut-off function $\eta \in C_c^\infty(B_R(x_0))$ with $\eta = 1$ on $B_{R/2}(x_0)$ and $|\nabla\eta| \leq 4/R$. We can now test in (1.1) with

$$\psi_\varepsilon = (u - \tilde{u})g_\varepsilon\eta^2.$$

Now once you have that $\phi(r)$ is a contraction you can conclude in a similar manner as before.

References

[1] Fabes E.B., Stroock D.W. (1989) A New Proof of Moser's Parabolic Harnack Inequality Using the old Ideas of Nash. In: Analysis and Continuum Mechanics. Springer, Berlin, Heidelberg. 3

[2] Struwe, Michael (2021) Lecture Notes on Elliptic Regularity Theory. 4